

On the Benjamin–Lighthill conjecture for water waves with vorticity

Vladimir Kozlov^a, Nikolay Kuznetsov^b and Evgeniy Lokharu^a

^a*Department of Mathematics, Linköping University, S-581 83 Linköping*

^b*Laboratory for Mathematical Modelling of Wave Phenomena,
Institute for Problems in Mechanical Engineering,
Russian Academy of Sciences*

V.O., Bol'shoy pr. 61, St Petersburg 199178, Russian Federation

E-mail: vlkoz@mai.liu.se / V. Kozlov; nikolay.g.kuznetsov@gmail.com /
N. Kuznetsov; evgeniy.lokharu@liu.se / E. Lokharu

Abstract

We consider the nonlinear problem of steady gravity-driven waves on the free surface of a two-dimensional flow of an incompressible fluid (say, water). The flow is assumed to be unidirectional of finite depth and the water motion is supposed to be rotational. Our aim is to verify the Benjamin–Lighthill conjecture for flows whose total head (Bernoulli's constant) is close to the critical one; the latter is determined by the vorticity distribution so that no horizontal shear flows exist for smaller values of the total head.

Originally, the conjecture was made about irrotational wave trains in order to describe them in terms of the parameters Q (rate of flow), R (total head/Bernoulli's constant) and S (flow force). Let r and s be dimensionless versions of R and S , respectively, for fixed Q , and let \mathcal{C} be the region in the (r, s) -plane whose cusped boundary $\partial\mathcal{C}$ represents all possible uniform streams; moreover, the part of $\partial\mathcal{C}$ corresponding to supercritical streams is included into \mathcal{C} , whereas the other part not. The Benjamin–Lighthill conjecture says that (a) each wave train is represented by a point of \mathcal{C} and (b) every point of \mathcal{C} corresponds to some wave train. In 2010–11, this form of the conjecture was proved by Kozlov and Kuznetsov for irrotational waves corresponding to nearcritical values of Bernoulli's constant.

Here, we modify the Benjamin–Lighthill conjecture to adapt it for rotational waves on unidirectional flows. Let ω be a vorticity distribution, then the corresponding cusped region \mathcal{C}_ω (its boundary represents all possible horizontal shear flows) must be truncated by the line $r = r_0$, where the constant r_0 defined by ω is finite for some vorticity distributions. Under the assumptions that ω is Lipschitz continuous and the problem's parameter r attains nearcritical values, we prove the following extended version of the conjecture. Namely, along with the assertions (a) and (b) formulated above we show that the correspondence between wave trains

and points in \mathcal{C}_ω is one-to-one. Our verification of the conjecture is based on the existence and uniqueness theorems for the problem with nearcritical values of r . These theorems also yield that there are two different parametrisations for each family of waves having their crests on a fixed vertical line.

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1 Introduction

We consider the two-dimensional nonlinear problem describing steady waves in a horizontal open channel of uniform rectangular cross-section. The corresponding motion of an inviscid incompressible heavy fluid, say water, occupying the channel is supposed to be rotational with a prescribed vorticity distribution. The reason for considering this mathematical model is the importance of vorticity for interaction of waves with currents (see the survey paper [24] by Peregrine) which commonly occurs in nature as is indicated by observations (see, for example, [25] and references cited therein). Indeed, an interesting phenomenon predicted in the framework of this problem is the formation of (possibly multiple) counter-currents separated one from the other by critical layers (see [7, 16, 26] and references cited therein).

However, the aim of the present article is to study the whole set of waves existing on *unidirectional* flows of constant depth that are close to the so-called critical shear flow. Unlike the irrotational case when the critical uniform stream is completely defined by its rate of flow and Bernoulli's constant, the vorticity distribution is essentially involved in the definition of the critical flow in the rotational case (see Section 2.1). Nevertheless, it occurs that the behaviour of waves with vorticity in some aspects is similar to that of irrotational ones. In particular, this concerns the topic of this paper—the Benjamin–Lighthill conjecture in the nearcritical regime. For its verification we establish that amplitudes and slopes of nearcritical waves are small, but the method used for this purpose is new and essentially differs from that applied in the irrotational case and based on harmonic analysis. Our approach provides a simple interpretation of waves existing for every nearcritical value of Bernoulli's constant. Namely, there are two different parametrisations for each family of waves having their crests on a fixed vertical line.

1.1 Statement of the Problem

Let an open channel of uniform rectangular cross-section be bounded below by a horizontal rigid bottom and let water occupying the channel be bounded above by a free surface not touching the bottom. In appropriate Cartesian coordinates (x, y) , the bottom coincides with the x -axis and gravity acts in the negative y -direction. We use the non-dimensional variables proposed by Keady and Norbury [11] (see also Appendix A in [16] for details of scaling); namely, lengths and velocities are scaled to $(Q^2/g)^{1/3}$ and $(Qg)^{1/3}$ respectively. Here Q and g are the dimensional quantities for the rate of flow and the gravity acceleration respectively, whereas $(Q^2/g)^{1/3}$ is the depth of the critical uniform stream in the irrotational case.

The steady water motion is supposed to be two-dimensional and rotational; the surface tension is neglected on the free surface of the water, where the pressure is constant. These assumptions and the fact that water is incompressible allow us to seek the velocity field in the form $(\psi_y, -\psi_x)$, where $\psi(x, y)$ is referred to as the *stream function*. The vorticity distribution ω is supposed to be

a prescribed Lipschitz function depending on ψ .

We choose the frame of reference so that the velocity field is time-independent as well as the unknown free-surface profile. The latter is assumed to be the graph of $y = \eta(x)$, $x \in \mathbb{R}$, where η is a positive continuous function, and so the longitudinal section of the water domain is $D = \{x \in \mathbb{R}, 0 < y < \eta(x)\}$. The following free-boundary problem for ψ and η which describes all kinds of waves has long been known (cf. [11]):

$$\psi_{xx} + \psi_{yy} + \omega(\psi) = 0, \quad (x, y) \in D; \quad (1.1)$$

$$\psi(x, 0) = 0, \quad x \in \mathbb{R}; \quad (1.2)$$

$$\psi(x, \eta(x)) = 1, \quad x \in \mathbb{R}; \quad (1.3)$$

$$|\nabla\psi(x, \eta(x))|^2 + 2\eta(x) = 3r, \quad x \in \mathbb{R}. \quad (1.4)$$

In condition (1.4) (Bernoulli's equation), r is a constant considered as the problem's parameter and referred to as Bernoulli's constant/the total head.

Initially, it is natural to impose rather weak assumptions on the unknown functions, namely, that $\psi \in C^1_{loc}(\bar{D})$ and η is a Lipschitz function on \mathbb{R} . This allows us to understand the boundary value problem (1.1)–(1.3) in a weak sense. Then the classical Schauder estimates are applicable because ω is a Lipschitz function. This implies that $\psi \in C^{2,\alpha}(D)$ for every $\alpha \in (0, 1)$, and so the problem (1.1)–(1.4) may be understood in the classical sense. Indeed, $\nabla\psi$ is continuous up to the boundary which yields that (1.4) is fulfilled in the classical sense.

Since we are going to study only unidirectional flows, it is assumed that the horizontal component of the velocity field has the same direction, say to the right, throughout the flow. This assumption results in the following additional condition:

$$\psi_y(x, y) > 0 \quad \text{for all } (x, y) \in \bar{D}. \quad (1.5)$$

In conclusion of this section, we recall that (ψ, η) is called a Stokes-wave solution of the problem (1.1)–(1.5) when η is a periodic function with a single crest per wavelength and symmetric about vertical lines going through crests, whereas $\psi(x, y)$ is a periodic function of x and its period is the same as that of η . Furthermore, a non-stream solution (ψ, η) is called a solitary-wave solution if it asymptotes some stream solution as $|x| \rightarrow \infty$ and η is symmetric about the vertical line going through the single crest. The class of stream solutions is analogous to uniform irrotational streams and is described below in Section 2.1.

1.2 Background

To compare the results obtained for rotational and irrotational waves, we recall what is known in both cases about the *whole class of steady* waves.

The first paper concerning this class in the *irrotational* case was [4]. In this paper published in 1954, Benjamin and Lighthill conjectured that the parameters Q , R , and S ‘probably determine the wave-train uniquely’. Here, Q is the volume rate of flow per unit span, R stands for the total head (Bernoulli's constant), and S is the flow force, and each of these parameters is a constant

of wave motion, that is, it does not depend on the coordinate measured along the horizontal bottom.

In order to formulate the irrotational Benjamin–Lighthill conjecture in precise terms we recall that both periodic and solitary waves bifurcate from uniform streams whose depths are defined by the following equation:

$$\left(\frac{Q}{\mathcal{H}}\right)^2 + 2g\mathcal{H} = 2R, \quad \mathcal{H} > 0, \quad (1.6)$$

where Q and R are given and g is the acceleration due to gravity. Namely, let Q be fixed, then for $R = R_c = \frac{3}{2}(Qg)^{2/3}$ there exists only one positive root of (1.6)—the double root $\mathcal{H}_c = (Q^2/g)^{1/3}$. The corresponding uniform stream is called *critical* because for $R < R_c$ there are no positive roots at all, whereas for $R > R_c$ the equation has two positive roots \mathcal{H}_- and \mathcal{H}_+ such that $\mathcal{H}_- < \mathcal{H}_c < \mathcal{H}_+$. The uniform stream whose depth is equal to \mathcal{H}_- (\mathcal{H}_+) is called *supercritical* (*subcritical*, respectively).

Using R_c and $S_c = \frac{3}{2}(Q^4g)^{1/3}$ (the critical values of R and S , respectively), two non-dimensional characteristics of flows supporting waves are defined as follows:

$$r = R/R_c \quad \text{and} \quad s = S/S_c.$$

We recall that r is usually considered as a given parameter in the problem of steady waves, whereas s depends on its solution as well as on r itself; see, for example, formula (2.4) in [3]. According to it, for $r > 1$ we have the following. The value of s corresponding to a supercritical stream is equal to $s_-(r) = (2 + \nu_-)/(3\nu_-^{1/3})$, whereas

$$s_+(r) = \frac{1}{3\nu_+^{1/3}} \left[2 + \nu_+ + \frac{1}{8} \left(\frac{\mathcal{H}_+}{\mathcal{H}_-} - 1 \right) \left(3 - \sqrt{1 + 8\nu_+} \right)^2 \right] \quad (1.7)$$

corresponds to a subcritical one. Here, $\nu_- > 1$ ($\nu_+ < 1$) is the larger (smaller, respectively) positive root of the following equation:

$$r = \frac{1 + 2\nu}{3\nu^{2/3}}.$$

It is worth mentioning that ν_- (ν_+) is the Froude number squared of the supercritical (subcritical, respectively) uniform stream. Both $s_-(r)$ and $s_+(r)$ monotonically increase with r , and $s_+(r) > s_-(r)$ for all $r > 1$, thus defining the cuspidal region

$$\mathcal{C} = \{(r, s) : 1 \leq r, \ s_-(r) \leq s < s_+(r)\},$$

see Figure 1 (a scaled version of Figure 2 in [4]). Here, the pairs $(r, s_-(r))$ corresponding to supercritical uniform streams are included because they also represent solitary-wave disturbances of these streams when these disturbances exist. In terms of \mathcal{C} the Benjamin–Lighthill conjecture is as follows:

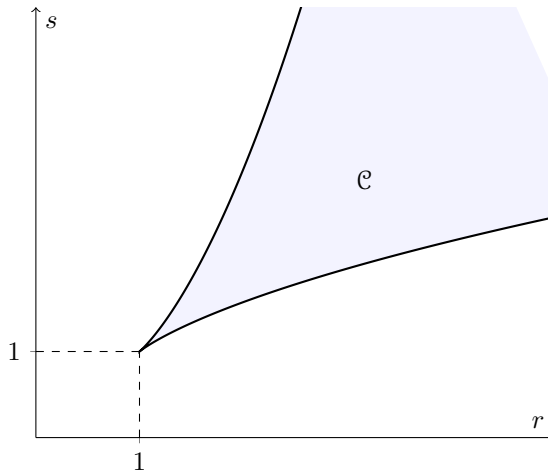


Figure 1: Possible values of (r, s) representing steady waves according to [4].

- (a) Every steady wave train is represented by a point in \mathcal{C} .
- (b) To every point in \mathcal{C} corresponds some steady wave train.
- (c) The correspondence between wave trains and points in \mathcal{C} is one-to-one.

In [4] (see also [3]), it was demonstrated that this conjecture is true in the framework of a one-dimensional approximate theory derived under the assumption that both $r - 1$ and $s - 1$ are small, that is, waves are long and of small amplitude.

Since no steady waves other than Stokes and solitary waves had been known when the conjecture was proposed, first results confirming assertion (a) were obtained for Stokes waves; namely, Keady and Norbury [11] (see also [3]) proved that Stokes waves are represented by inner points of \mathcal{C} . On the other hand, Ovsyannikov [23] supposed that $(r, s_-(r))$ uniquely determines a solitary wave, but Plotnikov [24] disproved this conjecture. He established that there are values of r for which at least two geometrically distinct solitary-wave profiles exist. Thus, assertion (c) of the Benjamin–Lighthill conjecture does not hold in its general form. It is worth mentioning that as early as 1974 Plotnikov’s result had been found numerically by Longuet-Higgins and Fenton [21].

Further results confirming the conjecture are as follows. Amick and Toland [2] showed that $r > 1$ for solitary waves (another proof of this fact was given by McLeod [22]). Subsequently, this fact was proved in [12] for all steady waves irrespective of the type (Stokes, solitary, whatever). On the other hand, numerical results obtained by Cokelet [6] show that assertion (b) is not true, at least for Stokes waves. There is a bound dividing \mathcal{C} so that *all* Stokes-wave solutions lie to the left of it. This bound is formed by points corresponding to waves that have greatest total head and flow force for a given wavelength; the corresponding line crosses the lower part of $\partial\mathcal{C}$ at some distance from the cusp vertex and approaches the curve (1.7) for large values of r (see Figure 1(a) in [11]). Other important results were obtained by Amick and Toland [2], who

combined the existence theorems for Stokes and solitary waves with the proof of the convergence of Stokes waves to solitary ones in the long-wave limit.

In the nearcritical case (r is close to one), the general structure of the whole set of irrotational waves was obtained in [13, 14]. It is as follows: only solitary and Stokes waves exist and they are parametrised by the depth at the crest which varies from the depth of the subcritical uniform flow to that of the solitary wave at its crest. This implies that assertions (a) and (b) of the conjecture are true in this case. Moreover, this hierarchy of waves is in agreement with the one-dimensional approximate theory investigated in [4] as well as with the bound obtained in [6].

Now, we turn to results for waves with vorticity. To the authors' knowledge, so far only Groves and Wahlén [9] studied small-amplitude Stokes and solitary waves using a unified approach based on the so-called spatial dynamics in which the horizontal coordinate plays the role of time. Their analysis demonstrates the existence of a continuous branch of wave solutions that consists of a solitary wave of elevation and a family of Stokes waves.

2 Main Results

The exact formulation of the Benjamin–Lighthill conjecture for waves with vorticity includes the same assertions (a)–(c) as in the irrotational case. However, the vorticity distribution must be involved in the definition of the corresponding cuspidal region (hence this region will be denoted by \mathcal{C}_ω in what follows) along with the total head r and the flow force invariant s . The latter is introduced below following the paper [11] by Keady and Norbury. Similarly to the irrotational case, we define \mathcal{C}_ω in terms of unidirectional horizontal shear flows which are analogous to uniform streams. Besides, *stream solutions* describing these flows have more complicated structure. For example, there are several definitions of the so-called Froude number for shear flows neither of which is universal like that for the irrotational uniform streams (see [27], p. 95, where these definitions are listed). Therefore, prior to the formulation of main results we outline basic properties of stream solutions and provide a classification of vorticity distributions based on these properties. This classification is also important when defining the cuspidal region \mathcal{C}_ω .

2.1 Stream Solutions

By a *stream* (shear-flow) solution we mean a pair $(u(y), d)$; u stands for the stream function instead of ψ and the constant depth of flow d replaces the wave profile η . Then problem (1.1)–(1.4) reduces the following one:

$$u'' + \omega(u) = 0 \quad \text{on } (0, d), \quad u(0) = 0, \quad u(d) = 1, \quad |u'(d)|^2 + 2d = 3r. \quad (2.1)$$

Here, the prime symbol denotes differentiation with respect to y . A detailed study of these solutions including those that describe flows with counter-currents is given in [15]. In particular, it is shown that the set of unidirectional solutions

of the first three relations (2.1) is parametrised by $\lambda = u'(0)$ which satisfies the inequality

$$\lambda \geq \lambda_0 = \sqrt{2 \max_{0 \leq \tau \leq 1} \Omega(\tau)}, \quad \text{where } \Omega(\tau) = \int_0^\tau \omega(t) dt.$$

This is a consequence of the following expressions for u and d (implicit and explicit respectively):

$$y = \int_0^u \frac{d\tau}{\sqrt{\lambda^2 - 2\Omega(\tau)}} \quad \text{and} \quad d = \int_0^1 \frac{d\tau}{\sqrt{\lambda^2 - 2\Omega(\tau)}}. \quad (2.2)$$

The function $d [= d(\lambda)]$ decreases strictly monotonically and tends to zero as $\lambda \rightarrow +\infty$ (see Figure 2), whereas $d_0 = \lim_{s \rightarrow \lambda_0+0} d(\lambda)$ can be finite or infinite depending on the behaviour of Ω on $[0, 1]$ (see below). It should be noted that the solution (2.2) is well defined for $\lambda = \lambda_0$ when $d_0 < +\infty$.

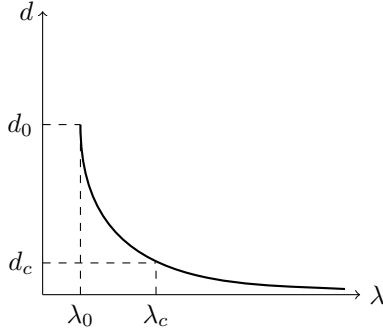


Figure 2: A sketch of the graph of $d(\lambda)$ in the case when $d_0 < \infty$.

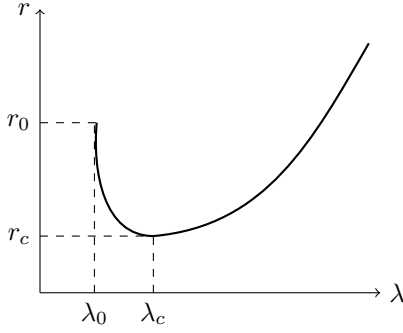


Figure 3: A sketch of the graph of $\mathcal{R}(\lambda)$ in the case when $d_0 < \infty$.

Furthermore, according to the last relation (2.1), the value of λ in formulae (2.2) must be determined from the equation

$$r = \mathcal{R}(\lambda), \quad \text{where } \mathcal{R}(\lambda) = [\lambda^2 - 2\Omega(1) + 2d(\lambda)]/3. \quad (2.3)$$

One comes to this conclusion by substituting the first expression (2.2) into the last relation (2.1). It is easy to check that the function $\mathcal{R}(\lambda)$ has only one minimum, say $r_c > 0$ attained at some $\lambda_c > \lambda_0$ (see Figure 3). Hence r_c is the critical value of r in the same sense as $r = 1$ is critical in the irrotational case, that is, no λ can be found from (2.3) when $\lambda < \lambda_c$. It is clear that

$$\int_0^1 \frac{d\tau}{[\lambda_c^2 - 2\Omega(\tau)]^{3/2}} = 1,$$

which, in particular, implies that $\lambda_c^2 - \lambda_0^2 \leq 1$.

If $d_0 = +\infty$, then for every $r > r_c$ the equation (2.3) has two solutions $\lambda_+(r)$ and $\lambda_-(r)$ such that $\lambda_0 < \lambda_+ < \lambda_c < \lambda_-$. By substituting λ_+ and λ_- into (2.2), one obtains two stream solutions, say (u_+, d_+) and (u_-, d_-) . However, if $d_0 < +\infty$, then both λ_+ and λ_- , and consequently the corresponding stream solutions satisfying inequality (1.5) exist only for $r \in (r_c, r_0)$, where $r_0 = \mathcal{R}(\lambda_0)$. It should be noted that $d_0 > d_+ > d_c > d_-$, where $d_c = d(\lambda_c)$. The shear flows described by (u_+, d_+) and (u_-, d_-) are analogous to the uniform sub- and supercritical flows respectively existing in the irrotational case.

It is worth mentioning that the values $d_+(r)$ and $d_-(r)$ provide important bounds for wave profiles on unidirectional rotational flows (see [17] for the proof). Namely, if $r \in (r_c, r_0)$, then the inequalities

$$d_-(r) < \eta(x) \quad \text{for all } x \in \mathbb{R} \quad \text{and} \quad \inf \eta < d_+(r) \leq \sup \eta$$

hold for all non-stream solutions. Moreover, the last inequality is strict provided $\sup \eta$ is attained somewhere. Finally, the problem (1.1)–(1.4) has no solution at all if $r < r_c$ and only the stream solution exists when $r = r_c$.

To analyse the dependence of d_0 on the vorticity distribution the following three options were considered in [15]:

(i)	$d_0 = +\infty$
(ii)	$d_0 < +\infty, u'(0) = 0, u'(d_0) \neq 0$
(iii)	$d_0 < +\infty, u'(d_0) = 0$

Thus, $d_0 < +\infty$ if either $u'(0) = 0$ or $u'(d_0) = 0$, and this classification can be reformulated in terms of the vorticity distribution as follows:

(i) $\max_{0 \leq p \leq 1} \Omega(p)$ is attained either at an inner point of $(0, 1)$ or at one (or both) of the end-points. In the latter case, either $\omega(1) = 0$ when $\Omega(1) > \Omega(p)$ for $p \in (0, 1)$ or $\omega(0) = 0$ when $\Omega(0) > \Omega(p)$ for $p \in (0, 1)$ (or both of these conditions hold simultaneously).

(ii) $\Omega(0) > \Omega(p)$ for $p \in (0, 1]$ and $\omega(0) < 0$.

(iii) $\Omega(p) < \Omega(1)$ for $p \in (0, 1)$ and $\omega(1) > 0$. Moreover, if $\Omega(1) = 0$, then $\omega(0) < 0$ and $\omega(1) > 0$ must hold simultaneously.

Conditions (i)–(iii) define three disjoint sets of vorticity distributions whose union gives the whole set of distributions that are continuous on $[0, 1]$.

2.2 Flow Force and the Cuspidal Region \mathcal{C}_ω

To the authors' knowledge, the flow force invariant s for rotational waves was introduced by Keady and Norbury [11]; up to a slight difference in the definition of Ω , their definition is as follows. Let (ψ, η) be a solution of the problem (1.1)–(1.4), then

$$s = s(\psi, \eta) = \left[r + \frac{2}{3} \Omega(1) \right] \eta(x) - \frac{1}{3} \left\{ \eta^2(x) - \int_0^{\eta(x)} [\psi_y^2 - \psi_x^2 - 2\Omega(\psi)] \, dy \right\}. \quad (2.4)$$

Indeed, using relations (1.1)–(1.4), it is straightforward to check that this expression is independent of x .

By $s_{\pm}(r)$ we denote the flow force corresponding to the stream solution $(u_{\pm}(y), d_{\pm})$, and so

$$s_{\pm}(r) = \left[r + \frac{2}{3}\Omega(1) \right] d_{\pm} - \frac{1}{3} \left\{ d_{\pm}^2 - \int_0^{d_{\pm}} [(u_{\pm})_y^2 - 2\Omega(u_{\pm})] dy \right\}.$$

It should be noted that the equation $s = s_+(r)$ ($s = s_-(r)$) gives the upper (lower respectively) curve bounding the cuspidal region \mathcal{C}_{ω} .

It occurs that the curve $s = s_-(r)$ always goes to infinity being defined for all $r \in [r_c, +\infty)$, whereas $s = s_+(r)$ goes to infinity only when $r_0 = +\infty$ or, what is the same, conditions (i) hold for the vorticity distribution. Thus, the cuspidal region \mathcal{C}_{ω} is similar to \mathcal{C} in this case. On the contrary, if ω satisfies either of conditions (ii) and (iii), then both d_0 and r_0 are finite, and so the curve $s = s_+(r)$ terminates at the point $(r_0, s_+(r_0))$. Hence the region \mathcal{C}_{ω} is bounded under these conditions and $\partial\mathcal{C}_{\omega}$ consists of two arcs and the segment that connects them and lies on the line $r = r_0$.

2.3 Definitions and Formulations

Since we are going to study waves only for nearcritical values of r , it is convenient to suppose that

$$r \leq \mathcal{R}([\lambda_c + \lambda_0]/2). \quad (2.5)$$

Also, the following notation will be used below:

$$\omega_0 = \max_{[0,1]} |\omega|, \quad \omega_1 = \omega_0 + \operatorname{ess\,sup}_{[0,1]} |\omega'|. \quad (2.6)$$

Now, we specify the problem to be investigated in this paper.

Definition. Let ω be a Lipschitz vorticity distribution. We say that (ψ, η) belonging to $C_{loc}^1(\bar{D}) \times W^{1,\infty}(\mathbb{R})$ is a solution of problem P_r^M for some $M > 0$ and $r > r_c$ ($r_c > 0$ is the critical value of Bernoulli's constant for ω), if the following conditions are fulfilled. The inequality $|\eta'(x)| \leq M$ is true a.e. on \mathbb{R} and (1.5) holds for ψ . The latter function satisfies the equation (1.1) in a weak sense, whereas the boundary conditions (1.2)–(1.4) are valid pointwise.

Let us turn to the formulation of main results concerning problem P_r^M . Our first theorem provides a complete description of the set of waves existing in the nearcritical regime; these are the family of Stokes waves and a solitary wave naturally parametrised by their heights at the crest (see Figure 4).

Theorem 2.1. *For any $M > 0$ there exists $r' \in (r_c, r_0)$ (it also depends on ω_1) such that the following assertions are true:*

- (I) *For every $r \in (r_c, r']$ problem P_r^M has one and only one solitary-wave solution $(\psi^{(s)}, \eta^{(s)})$ such that $\eta^{(s)}$ attains its maximum at $x = 0$ and is an even function. All other solitary-wave solutions are horizontal translations of $(\psi^{(s)}, \eta^{(s)})$.*

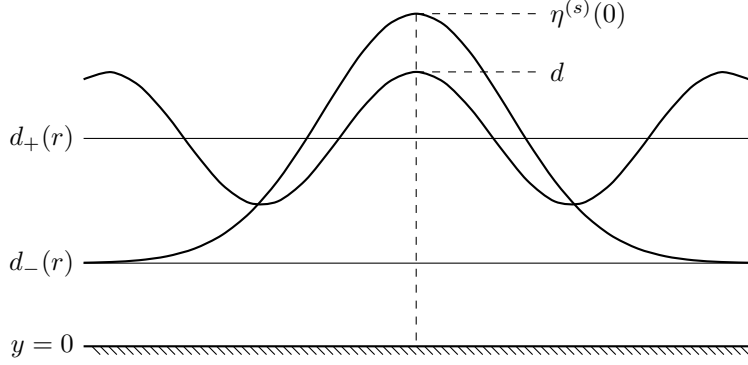


Figure 4: A sketch of the hierarchy of waves corresponding to a nearcritical value of r .

- (II) If (ψ, η) is a Stokes-wave solution of problem P_r^M with $r \in (r_c, r']$, then the following inequalities hold:

$$d_+ = d(\lambda_+(r)) < \max_{x \in \mathbb{R}} \eta(x) < \eta^{(s)}(0), \quad (2.7)$$

where $d(\lambda)$ is defined by the second formula (2.2) and $\lambda_+(r)$ is the smallest root of (2.3).

- (III) For every $r \in (r_c, r']$ and every $d \in (d_+, \eta^{(s)}(0))$ problem P_r^M has one and only one Stokes-wave solution (ψ, η) such that

$$\max_{x \in \mathbb{R}} \eta(x) = \eta(0) = d.$$

Thus, the family of Stokes waves with crests on the y -axis is parametrised by d . All other Stokes-wave solutions are horizontal translations of (ψ, η) .

- (IV) For every $r \in (r_c, r']$ all solutions of problem P_r^M are exhausted by those described in assertions (I) and (III) and stream solutions.

It should be mentioned that the left-hand side of inequality (2.7) was obtained in [17]; here we include it for the sake of completeness. The next assertion shows that all waves considered in Theorem 2.2 depend on the depth at the crest continuously, but the continuity is understood in a certain integral sense.

Theorem 2.2. *Given any $M > 0$, then there exists $r'' \in (r_c, r_0)$ (it also depends on ω_1) such that for any two solutions of problem P_r^M with $r \in (r_c, r'']$, say $(\psi^{(1)}, \eta^{(1)})$ and $(\psi^{(2)}, \eta^{(2)})$, the following inequality*

$$\begin{aligned} & \int_{\mathbb{R}} |\eta^{(1)}(x) - \eta^{(2)}(x)|^2 e^{-\theta|x-x_0|} dx \\ & \leq C \left[|\eta^{(1)}(x_0) - \eta^{(2)}(x_0)|^2 + |\eta_x^{(1)}(x_0) - \eta_x^{(2)}(x_0)|^2 \right] \end{aligned}$$

holds for any $x_0 \in \mathbb{R}$. The constants C and θ depend only ω_0 .

Theorems 2.1 and 2.2 are analogous to Theorems 2.1 and 2.2, respectively, proved in [13]. In the latter paper, the mentioned theorems provided the basis for verification of assertions (a) and (b) of the Benjamin–Lighthill conjecture for irrotational waves. Here, Theorems 2.1 and 2.2 serve for the same purpose for waves with vorticity (see Section 7 below), but our techniques used for establishing these theorems differ radically from those applied in [13].

Let $r' \in (r_c, r_0)$ be the number whose existence is established in Theorem 2.1, and let $r \in (r_c, r']$. By \mathcal{P}_r^M we denote the set of all solutions to problem P_r^M that have the following property. The second component η has a crest on the y -axis (see assertions (I) and (III) of Theorem 2.1 and Figure 4). Then formula (2.4) defines the following map:

$$\mathcal{P}_r^M \ni (\psi, \eta) \mapsto s \in [s_-(r), s_+(r)). \quad (2.8)$$

The last main result concerns this map; namely, the set \mathcal{P}_r^M is parametrised by the flow force s which is an alternative parametrisation to that considered in Theorem 2.1.

Theorem 2.3. *For any $M > 0$ there exists $r''' \in (r_c, r_0)$ (it also depends on ω_1) such that the map (2.8) is one-to-one for any $r \in (r_c, r''']$.*

This theorem yields that along with the parametrisation described in assertion (III) of Theorem 2.1 there is another parametrisation for the family of waves having their crests on the y -axis, namely, that in terms of s . Theorem 2.3 also allows us to verify assertion (c) of the Benjamin–Lighthill conjecture for rotational waves; irrotational ones are included as a particular case with zero vorticity.

It follows from our proofs of Theorems 2.2 and 2.3 that $r'' \geq r' = r'''$.

3 Reformulation of the Problem and an Auxiliary Sturm–Liouville Problem

In our proofs, an equivalent problem in a fixed strip $S = \mathbb{R} \times (0, 1)$ is used instead of the original problem (1.1)–(1.4) in the unknown domain D . The reformulation is possible for unidirectional flows and is based on the change of variables referred to as the partial hodograph transform (see Section 3.1) proposed by Dubreil-Jacotin [8] in 1934. In Sections 3.2 and 3.3, we investigate an auxiliary problem related to the linearized reformulated problem. This Sturm–Liouville problem was considered, for example, in [7], but the estimates of its eigenvalues and eigenfunctions (see Proposition 3.1 in Section 3.3), that are necessary for proving our main results, were not obtained so far.

3.1 Partial Hodograph Transform

In view of the boundary conditions (1.2), (1.3) and the inequality (1.5), putting $q = x$ and $p = \psi(x, y)$, we define a mapping

$$D \ni (x, y) \mapsto (p, q) \in S = \mathbb{R} \times (0, 1).$$

Let us treat the pair (q, p) as independent variables in S and consider y as the new unknown function for which $h(q, p)$ is the standard notation. A straightforward calculation shows that

$$h_q = -\frac{\psi_x}{\psi_y} \quad \text{and} \quad h_p = \frac{1}{\psi_y},$$

thus yielding that the problem (1.1)–(1.4) takes the following form

$$\left(\frac{h_q}{h_p}\right)_q - \frac{1}{2} \left(\frac{1+h_q^2}{h_p^2}\right)_p - \omega(p) = 0, \quad (q, p) \in S; \quad (3.1)$$

$$h(0, q) = 0, \quad q \in \mathbb{R}; \quad (3.2)$$

$$\frac{1+h_q^2}{h_p^2} = 3r - 2h, \quad p = 1, \quad q \in \mathbb{R} \quad (3.3)$$

in terms of new variables. On the other hand, using the formulae

$$\psi_x = -\frac{h_q}{h_p} \quad \text{and} \quad \psi_y = \frac{1}{h_p}, \quad (3.4)$$

one recovers the gradient of ψ ; that is, the velocity field $(\psi_y, -\psi_x)$ in \bar{D} . Besides, the equality $\eta(x) = h(x, 1)$, $x \in \mathbb{R}$ gives the free surface profile.

Let us emphasise two main advantages of the relations (3.1)–(3.3) comparing with (1.1)–(1.4) (to which they are equivalent under reasonable smoothness assumptions; see [7] for a detailed account). There is only one unknown function h and it is defined on the fixed strip \bar{S} .

In terms of new variables, a stream solution is a single function, say $H(p, \lambda)$, which is as follows:

$$H(p, \lambda) = \int_0^p \frac{d\tau}{\sqrt{\lambda^2 - 2\Omega(\tau)}}. \quad (3.5)$$

Here $\lambda \in (\lambda_c, \lambda_0]$ is the same parameter as in (2.2) and (2.3). It is clear that $d_+(r) = H(1, \lambda_+(r))$.

3.2 Auxiliary Sturm–Liouville Problem

It is straightforward to check that the equation obtained by linearization of (3.1) near the stream solution H involves the operator $-\partial_p (H_p^{-3} \partial_p)$. This is the reason to consider the following Sturm–Liouville problem:

$$-\left(\frac{\phi_p}{H_p^3}\right)_p = \mu \frac{\phi}{H_p}, \quad p \in (0, 1); \quad (3.6)$$

$$\phi_p(1) = H_p^3(1)\phi(1); \quad (3.7)$$

$$\phi(0) = 0. \quad (3.8)$$

Here μ is the spectral parameter, but one has to keep in mind that the eigenvalues and eigenfunctions of this problem depend also on λ because H depends

on it. In our investigation of the problem (3.1)–(3.3), the problem (3.6)–(3.8) plays an essential role. Let us denote by $\{\mu_k\}_{k=0}^{+\infty}$ the sequence of its eigenvalues all of which are simple, that is,

$$\mu_0 < \mu_1 < \cdots < \mu_n < \cdots;$$

here μ_0 may be either positive or negative.

3.2.1 A Preliminary Estimate of μ_1

Let

$$\mathfrak{M} = \max_{p \in [0,1]} H_p = (\lambda^2 - \lambda_0^2)^{-1/2} \text{ and } \mathfrak{m} = \min_{p \in [0,1]} H_p = [\lambda^2 - 2 \min_{\tau \in [0,1]} \Omega(\tau)]^{-1/2},$$

then

$$\mu_1 \geq \pi^2 \frac{\mathfrak{m}}{\mathfrak{M}^3}. \quad (3.9)$$

Indeed, it is well known that the smallest eigenvalue of the operator $-d^2/dp^2$ on $(0, 1)$ with the Dirichlet boundary conditions is

$$\inf_{\phi \in W_0^{1,2}(0,1)} \frac{\int_0^1 \phi_p^2 dp}{\int_0^1 \phi^2 dp} = \pi^2.$$

Since the fundamental eigenfunction ϕ_0 of the problem (3.6)–(3.8) is non-negative irrespective of the sign of the corresponding eigenvalue μ_0 , the eigenfunction ϕ_1 has exactly one zero, say $p_* \in (0, 1)$. Therefore, we have

$$\pi^2 \leq p_*^2 \frac{\int_0^{p_*} \phi_{1p}^2 dp}{\int_0^{p_*} \phi_{1p}^2 dp} \leq \frac{\mathfrak{M}^3}{\mathfrak{m}} \frac{\int_0^{p_*} \phi_{1p}^2 H_p^{-3} dp}{\int_0^{p_*} \phi_{1p}^2 H_p^{-1} dp} \leq \frac{\mathfrak{M}^3}{\mathfrak{m}} \mu_1,$$

which yields inequality (3.9).

3.2.2 Properties of μ_0 and ϕ_0

Let us consider the initial value problem

$$-(H_p^{-3} V_p)_p = \mu H_p^{-1} V, \quad V(0) = 0, \quad V_p(0) = 1, \quad (3.10)$$

depending on the parameters μ and λ (the latter is involved through H), and so we will write $V(p; \lambda, \mu)$ when necessary. It should be noted that V is a monotone function of $p \in [0, 1]$, and it analytically depends on $\lambda > \lambda_0$ and $\mu \in \mathbb{R}$. In terms of V , the eigenvalue $\mu_0 = \mu_0(\lambda)$ of the problem (3.6)–(3.8) is equal to the least root of the equation $\sigma(\lambda, \mu) = 0$, where

$$\sigma(\lambda, \mu) = [H_p(1, \lambda)]^{-3} V_p(1; \lambda, \mu) - V(1; \lambda, \mu),$$

which depends on $\lambda > \lambda_0$ and $\mu \in \mathbb{R}$ analytically. Moreover, the eigenfunction ϕ_0 corresponding to μ_0 is equal to V solving (3.10) with $\mu = \mu_0$.

In order to find the derivative σ_μ , it is convenient to differentiate the first equality (3.10) with respect to μ , then multiply the result by V and integrate over $(0, 1)$. Then we obtain after integration by parts

$$\sigma_\mu(\lambda, \mu) = \frac{-1}{V(1; \lambda, \mu)} \int_0^1 \frac{V^2(p; \lambda, \mu)}{H_p(p, \lambda)} dp. \quad (3.11)$$

Hence σ_μ is always negative which implies that $\sigma(\lambda, \mu) > 0$ when $\mu < \mu_0(\lambda)$.

Furthermore, if $\mu = 0$, then $V(p; \lambda, 0) = -\lambda^2 \partial_\lambda H(p; \lambda)$, and so

$$\sigma(\lambda, 0) = \lambda^2 \partial_\lambda \left[\frac{1}{2H_p^2(1; \lambda)} + H(1; \lambda) \right] = \frac{3\lambda^2}{2} \mathcal{R}'(\lambda), \quad (3.12)$$

where the last equality is a consequence of the definitions of H and \mathcal{R} (see (3.5) and (2.3) respectively). Since $\mathcal{R}'(\lambda)$ is negative for $\lambda < \lambda_c$ and positive for $\lambda > \lambda_c$ (see Figure 2), it follows from (3.12) and (3.11) that $\mu_0(\lambda)$ is positive (negative) for $\lambda > \lambda_c$ ($\lambda < \lambda_c$ respectively) and $\mu_0(\lambda_c) = 0$.

Let us estimate $|\mu_0(\lambda)|$ for $\lambda < \lambda_c$ or, what is the same, $\mu_0(\lambda) < 0$. We have

$$\mu_0 = \min_{v \in W_0^{1,2}(0,1)} \frac{\int_0^1 v^2 H_p^{-3} dp - v^2(1)}{\int_0^1 v^2 H_p^{-1} dp} \geq \min_{v \in W_0^{1,2}(0,1)} \frac{\mathfrak{M}^{-3} \int_0^1 v^2 dp - v^2(1)}{\mathfrak{m}^{-1} \int_0^1 v^2 dp},$$

where \mathfrak{M} and \mathfrak{m} are the same as in (3.9). To find the last minimum one has to find the fundamental eigenvalue of the problem that has the same form as (3.6)–(3.8), but its constant coefficients are obtained by changing H_p to \mathfrak{M} . Then the minimum is equal to $-\kappa^2 \mathfrak{m} \mathfrak{M}^{-3}$, where $\kappa = \kappa(\mathfrak{M})$ is the root of the following equation:

$$\frac{\sinh \kappa}{\kappa \cosh \kappa} = \frac{1}{\mathfrak{M}^3}.$$

Since $\mathfrak{M}^3 \geq \int_0^1 H_p^3 dp > 1$, this equation is uniquely solvable and we arrive at the following estimate:

$$|\mu_0| \leq \kappa^2 \mathfrak{M}^{-2}. \quad (3.13)$$

Using the variational formulation of the problem (3.1)–(3.3), one obtains that the right-hand side of the last inequality is a monotone function of \mathfrak{M} .

Now, we turn to estimates of V and V_p . Integrating the equation (3.10) and using the second boundary condition, we get that

$$V_p(p) = \frac{H_p^3(p)}{\lambda^3} + |\mu| H_p^3(p) \int_0^p \frac{V(p')}{H_p(p')} dp', \quad (3.14)$$

which gives the following lower estimates:

$$V(p) \geq \frac{p \mathfrak{m}^3}{\lambda^3}, \quad V_p(p) \geq \frac{\mathfrak{m}^3}{\lambda^3}. \quad (3.15)$$

In order to obtain an upper estimate for V , we integrate (3.14), thus reducing it to an integral equation with a positive operator. Its upper solution gives an

estimate of V , and one obtains such a solution from the problem (3.6)–(3.8) with H_p^{-3} and H_p^{-1} changed to \mathfrak{M}^{-3} and \mathfrak{m}^{-1} , respectively. This leads to the following inequality:

$$V \leq \frac{\sinh(\theta p)}{\theta}, \quad \text{where } \theta^2 = |\mu| \frac{\mathfrak{M}^3}{\mathfrak{m}}. \quad (3.16)$$

Combining this and (3.14), we get that

$$V_p \leq \frac{\mathfrak{M}^3}{\lambda^3} + |\mu| \frac{\mathfrak{M}^3}{\mathfrak{m}} \int_0^p \frac{\sinh(\theta p)}{\theta} dp. \quad (3.17)$$

Let us estimate $\mu'_0(\lambda)$, for which purpose we differentiate the equation (3.10) with respect to λ . The result multiplied by V we integrate over $(0, 1)$ and after integration by parts arrive at

$$\sigma_\lambda(\lambda, \mu) = \frac{1}{V(1; \lambda, \mu)} \left(\int_0^1 V_p^2 \partial_\lambda H_p^{-3} dp - \mu \int_0^1 V^2 \partial_\lambda H_p^{-1} dp \right). \quad (3.18)$$

Since $\mu'_0(\lambda) = -\sigma_\lambda(\lambda, \mu(\lambda))/\sigma_\mu(\lambda, \mu(\lambda))$, relations (3.11) and (3.18) yield that $\mu(\lambda)$ is a Lipschitz function on every interval separated from λ_0 .

3.3 Estimates for Nearcritical Values of r

In this section, we refine the estimates obtained in Sections 3.1 and 3.2 by applying the assumption (3.16), which means that $\lambda \in [(\lambda_0 + \lambda_c)/2, \lambda_c]$.

Proposition 3.1. *If $\lambda \in [(\lambda_0 + \lambda_c)/2, \lambda_c]$, then the following inequalities are valid:*

- (i) $\mu_0(\lambda) < 0$ and $|\mu_0(\lambda)| \leq C_1 |\lambda - \lambda_c|$,
- (ii) $\mu_1(\lambda) \geq C_2$,
- (iii) $C_3 p \leq V(p) \leq C_4 p$ and $C_5 \leq V_p(p) \leq C_6$ for $p \in [0, 1]$.

The positive constants C_1, \dots, C_6 depend only on ω_0 (see the first formula (2.6) for its definition).

Proof. It is easy to see that the right-hand side terms in (3.9), (3.13) and (3.15)–(3.17) depend monotonically on both \mathfrak{m} and \mathfrak{M} . Therefore, these estimates imply that the inequalities listed in items (i)–(iii) follow from the inequalities

$$C'(\omega_0) \leq \lambda_c \leq C''(\omega_0) \quad (3.19)$$

provided the estimates

$$\mathfrak{m} \geq \mathfrak{m}_* \quad \text{and} \quad \mathfrak{M} \leq \mathfrak{M}_* \quad (3.20)$$

hold with positive constants \mathfrak{m}_* and \mathfrak{M}_* depending only on ω_0 .

The definition of λ_c implies that $\lambda_c^2 - \lambda_0^2 \leq 1$, and so we have that

$$\mathfrak{m} \geq \frac{1}{\sqrt{\lambda_c^2 - \lambda_0^2 + \lambda_0^2 - 2 \min_{\tau \in [0, 1]} \Omega(\tau)}} \geq \frac{1}{\sqrt{1 + 4\omega_0}},$$

which gives the first inequality (3.20). Furthermore, we see that

$$\mathfrak{M} \leq \frac{2}{\sqrt{\lambda_c^2 - \lambda_0^2}}.$$

Hence it is sufficient to estimate $\lambda_c^2 - \lambda_0^2$ from below in order to obtain the second inequality (3.20). Let τ_0 be such that $2\Omega(\tau_0) = \lambda_0^2$, then we have

$$\int_0^1 (\lambda_c^2 - \lambda_0^2 + 2\omega_0|\tau - \tau_0|)^{-3/2} d\tau \leq \int_0^1 [\lambda_c^2 - 2\Omega(\tau)]^{-3/2} d\tau = 1.$$

Evaluating the integral on the left-hand side, we get the inequality

$$\frac{1}{\omega_0} \left[2(\lambda_c^2 - \lambda_0^2)^{-1/2} - (\lambda_c^2 - \lambda_0^2 + 2\omega_0(1 - \tau_0))^{-1/2} - (\lambda_c^2 - \lambda_0^2 + 2\omega_0\tau_0)^{-1/2} \right] \leq 1.$$

It implies that $(\lambda_c^2 - \lambda_0^2)^{-1/2} - (\lambda_c^2 - \lambda_0^2 + \omega_0)^{-1/2} \leq \omega_0$, thus yielding

$$1 \leq 2(\lambda_c^2 - \lambda_0^2 + \omega_0) \sqrt{\lambda_c^2 - \lambda_0^2}.$$

Hence either $1/4 \leq (\lambda_c^2 - \lambda_0^2)^{3/2}$ or $1/4 \leq \omega_0(\lambda_c^2 - \lambda_0^2)^{1/2}$, and so

$$\frac{1}{4^{2/3} + (4\omega_0)^2} \leq \lambda_c^2 - \lambda_0^2,$$

which gives the second inequality (3.20).

Furthermore, combining the last inequality and $\lambda_c^2 - \lambda_0^2 \leq 1$, we obtain (3.19) with

$$C' = \sqrt{\lambda_0^2 + [4^{2/3} + (4\omega_0)^2]^{-1}} \quad \text{and} \quad C'' = \sqrt{1 + \lambda_0^2}.$$

This completes the proof of the proposition. \square

4 Properties of Solutions of Problem P_r^M

In this section, we study the following properties of solutions. First, we obtain a lower bound for $\tilde{\eta} = \inf_{x \in \mathbb{R}} \eta(x)$, which is similar to the first assertion of Theorem 1, [17], where the inequality $\tilde{\eta} \geq d_-(r)$ was proved. (We cannot use that theorem here because it was established under stronger assumptions than those imposed in problem P_r^M .) Then we derive a uniform bound for $\nabla\psi$ that depends only on ω_0 provided Bernoulli's constant is separated from r_0 . Moreover, bounds are obtained for the Hölder norms of both ψ and η . Finally, it is shown that the smallness of $\eta - d_+(r)$ depends on the difference $r - r_c$.

In what follows, we use different local estimates for solutions of problem P_r^M near the free surface, for which purpose we have to ascertain how thick is the domain D .

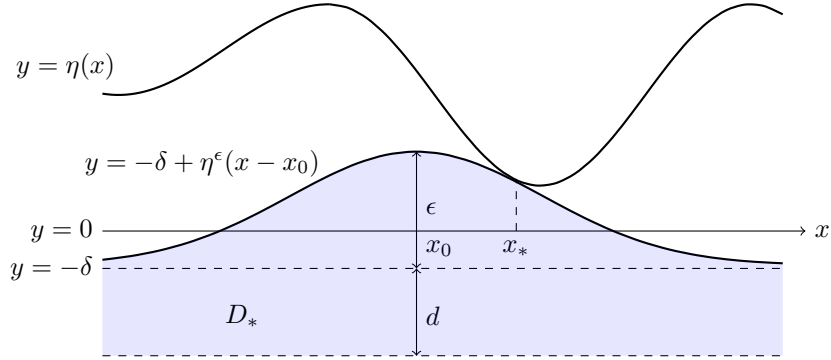


Figure 5: A sketch of the domain D_* .

Lemma 4.1. *Let (ψ, η) be a solution of problem P_r^M , then the following inequality holds: $\check{\eta} \geq \min \{ (6r)^{-1/2}, (2\omega_0)^{-1/2} \}$.*

Proof. Let $d > 0$ and $\epsilon \in (0, 1)$ be given numbers, and let $\eta_\epsilon(x) = \epsilon e^{-x^2}$. In the domain $D_{d,\epsilon} = \{(x, y) \in \mathbb{R}^2 : 0 < y < d + \eta_\epsilon(x)\}$, we consider the auxiliary boundary value problem

$$\nabla^2 \Psi + \omega_0 = 0 \text{ in } D_{d,\epsilon}, \quad \Psi(x, 0) = 0, \quad \Psi(x, d + \eta_\epsilon(x)) = 1,$$

whose bounded solution we denote by $\psi_{d,\epsilon}$. Let us show that

$$\|\psi_{d,\epsilon} - u_{d,\epsilon}\|_{C^1(D_{d,\epsilon})} \leq \epsilon C_{d,\omega_0}, \quad (4.1)$$

where $u_{d,\epsilon}(x, y) = u_d(dy/[d + \eta_\epsilon(x)])$ and $u_d(y) = y[d^{-1} + \omega_0(d - y)/2]$. It is clear that the last function satisfies the following relations:

$$u_d'' + \omega_0 = 0 \text{ on } (0, 1), \quad u_d(0) = 0, \quad u_d(1) = 1.$$

Moreover, it is an increasing function on $[0, d]$ when $d < d_* = \sqrt{2/\omega_0}$. In order to prove (4.1), we note that $u = \psi_{d,\epsilon} - u_{d,\epsilon}$ solves the homogeneous Dirichlet problem for the following equation:

$$\nabla^2 u = \epsilon \left\{ \left[\frac{2dx e^{-x^2}}{d + \epsilon e^{-x^2}} u_d' \left(\frac{dy}{d + \eta_\epsilon(x)} \right) \right]_x - \frac{\omega_0 e^{-x^2} [2d + \eta_\epsilon(x)]}{[d + \eta_\epsilon(x)]^2} \right\} \text{ in } D_{d,\epsilon}.$$

This yields that $\|u\|_{W^{2,1}(D_{d,\epsilon})} \leq \epsilon C_{d,\omega_0}$, from which (4.1) follows by virtue of local estimates.

Now we turn to the lower bound for $\check{\eta}$ and consider two cases. First, let us assume that $\check{\eta} = 0$. Since η does not vanish identically, for some $d \in (0, d_*)$ and $\delta > 0$ there exists $\epsilon \in (\delta, 2\delta)$ and $x_0, x_* \in \mathbb{R}$ (see Figure 5) such that

$$\eta(x) \geq -\delta + \eta_\epsilon(x - x_0) \quad \text{and} \quad \eta(x_*) = -\delta + \eta_\epsilon(x_* - x_0).$$

Let us apply the maximum principle to the superharmonic function

$$U(x, y) = \psi_{d, \epsilon}(x - x_0, y + d + \delta) - \psi(x, y)$$

in the domain

$$D_* = \{(x, y) : -Q < x - x_0 < Q, \quad 0 < y < -\delta + \eta_\epsilon(x - x_0)\},$$

where $Q > 0$ is such that $\eta_\epsilon(Q) = \delta$. Since U is positive in this domain and $U(x_*, -\delta + \eta_\epsilon(x_* - x_0))$ vanishes, we obtain that $\partial_n U(x_*, -\delta + \eta_\epsilon(x_* - x_0)) < 0$, and so $u'_d(d) \leq \sqrt{3r} + O(\delta)$. Letting $\delta \rightarrow 0$ in this inequality, we arrive at $u'_d(d) \leq \sqrt{3r}$, which is impossible when d is sufficiently small. The obtained contradiction shows that $\tilde{\eta} > 0$.

Let us assume that the positive $\tilde{\eta}$ is less than $(2\omega_0)^{-1/2}$, because otherwise the required inequality is obviously true. To keep the same notation as in the previous case we put $d = \tilde{\eta}$. Then for every $\delta \in (0, d)$ there exists $\epsilon \in (\delta, 2\delta)$ and $x_0, x_* \in \mathbb{R}$ such that

$$\eta(x) \geq d - \delta + \eta_\epsilon(x - x_0) \quad \text{and} \quad \eta(x_*) = d - \delta + \eta_\epsilon(x_* - x_0).$$

Now we apply the maximum principle to the superharmonic function

$$U(x, y) = \psi_{d, \epsilon}(x - x_0, y + \delta) - \psi(x, y)$$

in the domain $\{(x, y) : -\infty < x < +\infty, \quad 0 < y < d - \delta + \eta_\epsilon(x - x_0)\}$, thus concluding (similarly to the previous case) that $\partial_n U(x_*, d - \delta + \eta_\epsilon(x_* - x_0)) < 0$. Therefore,

$$u'_d(d) = \tilde{\eta}^{-1} - \omega_0 \tilde{\eta} / 2 \leq \sqrt{3r - \tilde{\eta}},$$

and so $\tilde{\eta} \geq \min \{(6r)^{-1/2}, (2\omega_0)^{-1/2}\}$. It should be noted that this inequality for $\tilde{\eta}$ also holds when $\tilde{\eta} \geq d_0$, which completes the proof. \square

4.1 Uniform Bound for Velocity Field

Here we give a uniform bound for $\nabla \psi$, and the main difficulty is the nonlinear term in equation (1.1) which does not allow us to apply the standard maximum principle for elliptic equations.

Proposition 4.1. *For every $R > r_c$ there exists a constant $C(\omega_0, R)$ (it does not depend on other parameters) such that the inequality $|\nabla \psi(x, y)| \leq C(\omega_0, R)$ holds for all $(x, y) \in \bar{D}$ provided (ψ, η) solves problem P_r^M with $r \in (r_c, R]$.*

Proof. Let $D_t = \{(x, y) : t \leq x \leq t + 1, 0 \leq y \leq \eta(x)\}$ for an arbitrary $t \in \mathbb{R}$. Our first aim is to show that

$$\sup_{t \in \mathbb{R}} \int_{D_t} |\nabla \psi|^2 dx dy \leq C(\omega_0, M, r), \quad (4.2)$$

where the constant $C(\omega_0, M, r)$ does not depend on other parameters. For this purpose we multiply the equation (1.1) by ψ , integrate the result over D_t and apply the first Green's formula, thus obtaining

$$\begin{aligned} \int_{D_t} |\nabla \psi|^2 dx dy &= \int_{D_t} \omega(\psi) \psi dx dy + \int_t^{t+1} \nabla \psi \cdot (-\eta'(x), 1) dx \\ &+ \int_0^{\eta(t+1)} \psi_x(t+1, y) \psi(t+1, y) dy - \int_0^{\eta(t)} \psi_x(t, y) \psi(t, y) dy. \end{aligned} \quad (4.3)$$

Let us consider each of the four integrals on the right-hand side, say I_1, I_2, I_3, I_4 .

Since the values of ψ belong to $[0, 1]$ according to (1.5), the definition of ω_0 implies that $|I_1| \leq 3r\omega_0/2$ for all $t \in \mathbb{R}$ because $\eta(x) \leq 3r/2$ for all $x \in \mathbb{R}$. Furthermore, the boundary condition (1.3) yields that

$$\psi_x(x, \eta(x)) = -\eta'(x) \psi_y(x, \eta(x)),$$

and so the Bernoulli equation (1.4) gives

$$0 < I_2 = \int_t^{t+1} \sqrt{[3r - 2\eta(x)][1 + \eta'^2(x)]} dx \leq \sqrt{3r(1 + M^2)} \quad \text{for all } t \in \mathbb{R}.$$

Our estimates of $|I_3|$ and $|I_4|$ are based on the equality

$$\int_0^{\eta(t)} \psi_x(t, y) \psi(t, y) dy = 2^{-1} \frac{d}{dt} \left[\int_0^{\eta(t)} \psi^2(t, y) dy - \eta(t) \right],$$

which allows us to estimate

$$\left| \int_{\mathbb{R}} e^{-|t-\tilde{t}|} d\tilde{t} \int_0^{\eta(\tilde{t})} \psi_x(\tilde{t}, y) \psi(\tilde{t}, y) dy \right| \quad \text{for all } t \in \mathbb{R}.$$

Indeed, multiplying the right-hand side of the previous formula by the corresponding exponential and integrating over \mathbb{R} , we get after integration by parts that the last integral is less than or equal to $3r$.

Thus, it follows from (4.3) and the obtained inequalities that

$$\int_{\mathbb{R}} e^{-|t-\tilde{t}|} d\tilde{t} \int_{D_{\tilde{t}}} |\nabla \psi|^2 dx dy \leq C(\omega_0, M, r) \quad \text{for all } t \in \mathbb{R}.$$

After changing the order of integration, this gives

$$\int_D |\nabla \psi|^2 e^{-|x-t|} dx dy \leq C(\omega_0, M, r) \quad \text{for all } t \in \mathbb{R},$$

which, in its turn, yields (4.2).

To complete the proof we apply some local estimates. First, we concentrate on interior estimates and estimates near the bottom for which purpose we use only the equation (1.1) and the boundary condition (1.2).

The Schauder interior estimates (see [10], Theorem 6.2) and the inequality (4.2) imply that $|\nabla\psi|$ is bounded pointwise by a constant $C(\omega_1, \epsilon)$ in the domain D_ϵ whose points are distant from ∂D not less than $\epsilon = \tilde{\eta}/3$. Besides, combining local estimates near a smooth boundary (see [10], Corollary 8.36) and the inequality (4.2), one obtains that $|\nabla\psi|$ is bounded pointwise by a constant $C(\omega_0, \epsilon)$ in the ϵ -neighbourhood of the bottom $\{x \in \mathbb{R}, y = 0\}$. Furthermore, Lemma 4.1 shows that $\epsilon = \tilde{\eta}/3$ is greater than a positive constant depending only on ω_0 and R .

To estimate $\nabla\psi$ near the free surface we use Theorem 8.25, [10], for solutions of the equation $\nabla^2\psi^* + \omega'(\psi)\psi^* = 0$, thus obtaining

$$\sup_{B_a(Z_0)} \psi_m^* \leq C(\omega_1) a^{-1} \|\psi_m^*\|_{L^2(B_{2a}(Z_0))}. \quad (4.4)$$

Here ψ^* stands for either of the first derivatives of ψ , $a > 0$ is fixed and Z_0 is an arbitrary point of D , $B_a(Z_0)$ denotes the open circle of radius a centred at Z_0 , $m = \sup_{\partial D \cap B_{2a}(Z_0)} \psi^*$ and

$$\psi_m^*(x, y) = \begin{cases} \max\{\psi^*(x, y), m\} & \text{when } (x, y) \in D, \\ m & \text{when } (x, y) \notin D. \end{cases}$$

Applying (4.4) in an arbitrary ball centred at the free surface with $a = \epsilon = \tilde{\eta}/3$, we note that the Bernoulli equation (1.4) implies that $m \leq \sqrt{3r}$. Combining (4.2) and (4.4), we find that ψ^* is bounded from above by $C(\omega_1, M, r)$ in the ϵ -neighbourhood of the free surface. Since the same argument is valid for $-\psi^*$, we get that

$$|\nabla\psi(x, y)| \leq C(\omega_1, M, r) \quad \text{for all } (x, y) \in D.$$

In order to prove that a similar estimate holds with a constant independent of M we consider the function

$$P(x, y) = -3r + |\nabla\psi|^2 + 2y - 2\Omega(\psi).$$

A direct calculation gives that $\nabla^2 P \geq \omega_0^2$ in D , and so $P^* = P + \omega_0^2 y^2/2$ is a subharmonic function in D . Since P^* is bounded, the maximum principle (see [5], Theorem 1.4) is applicable. Hence the supremum of P^* is attained on the boundary. On the free surface, P^* is bounded by $2\omega_0 + 3\omega_0^2 r^2$, whereas there exists a constant $C(\omega_0, R)$ such that $P^* \leq C(\omega_0, R)$ on the bottom. Therefore, $P^* \leq C(\omega_0, R)$ throughout D , which proves the proposition when one takes into account the definition of P . \square

4.2 Bounds for Solutions of P_r^M in the Nearcritical Case

By the definition of problem P_r^M , the slope of η is bounded, but this restriction does not prevent that stagnation points might be present on η . However, if the Bernoulli constant r is sufficiently close to its critical value r_c , then any solution of P_r^M is of small amplitude, and so there are no stagnation points on η . The following assertion deals with both these properties.

Theorem 4.1. *Let M be positive and r is subject to condition (2.5). Then the following assertions are true.*

- (a) *For every $\epsilon > 0$ there exists a constant $r'(\epsilon, M, \omega_0) > r_c$ such that the inequality $\sup_{x \in \mathbb{R}} |\eta(x) - d_+(r)| < \epsilon$ is valid for the second component of a solution of problem P_r^M with $r \in (r_c, r']$. Here $d_+(r)$ is the quantity defined in Section 2.1.*
- (b) *There exist $r''(M, \omega_1) > r_c$ and $\delta(M, \omega_1) > 0$ such that the inequality $\psi_y > \delta(M, \omega_1)$ is fulfilled in D for the first component of a solution of problem P_r^M with $r \in (r_c, r'']$.*

Proof. To prove (a), it is reasonable to use $w(q, p) = h(q, p) - H(p)$ because $\eta(x) = h(x, 1)$, $x \in \mathbb{R}$, and $d_+(r) = H(1, \lambda_+(r))$; here h is a solution of the problem (3.1)–(3.3) in $S = \{(q, p) : q \in \mathbb{R}; p \in (0, 1)\}$ and H is defined by formula (3.5). Relations (3.4) between the derivatives of ψ and h yield that h , h_p^{-1} and $f = h_q h_p^{-1}$ are bounded by a constant depending on ω_0 and R (see Proposition 4.1).

Let ϕ_0 be the eigenfunction of the problem (3.6)–(3.8) corresponding to μ_0 . If $H = H(p; \lambda_+(r))$, then ϕ_0 is negative (see assertion (i) of Proposition 3.1). Putting

$$w_0(q) = \int_0^1 w(q, p) \phi_0(p) H_p^{-1}(p) dp \quad \text{and} \quad f_0(q) = \int_0^1 f(q, p) \phi_0(p) dp,$$

we get from (3.1)–(3.3)

$$\int_0^1 \left[\frac{H_p^3 f^2}{2} + \frac{(2h_p + H_p)w_p^2}{2h_p^2} \right] \frac{\phi_0'}{H_p^3} dp = \mu_0 w_0 - f_0'.$$

Multiplying this identity by $e^{-\vartheta|q-q_0|}$ with some $\vartheta > 0$ and $q_0 \in \mathbb{R}$, and integrating the result over \mathbb{R} , we estimate both integrals on the right-hand side, thus obtaining

$$\begin{aligned} & \int_S \left[\frac{H_p^3 f^2}{2} + \frac{(2h_p + H_p)w_p^2}{2h_p^2} \right] \frac{\phi_0'}{H_p^3} e^{-\vartheta|q-q_0|} dq dp \\ & \leq |\mu_0| \int_{-\infty}^{+\infty} |w_0| e^{-\vartheta|q-q_0|} dq + C_4(\omega_0) \vartheta \int_S |f| e^{-\vartheta|q-q_0|} dq. \end{aligned} \quad (4.5)$$

Here C_4 is the same constant as in Proposition 3.1, and it arises after integration by parts in the second integral. Since w_0 and f are bounded on \bar{S} , both integrals on the right-hand side are convergent. Therefore, the first term is bounded by $C(\omega_0)|\mu_0|/\vartheta$, whereas the second one is less than or equal to

$$C(\omega_0) \sqrt{\vartheta} \left[\int_S f^2 e^{-\vartheta|q-q_0|} dq dp \right]^{1/2},$$

which is a consequence of the Schwarz inequality. Putting $\vartheta = \sqrt{|\mu_0|}$, we get from (4.5)

$$I^2 - C(\omega_0)|\mu_0|^{1/4}I - C(\omega_0)\sqrt{|\mu_0|} \leq 0, \quad \text{where } I = \left[\int_S f^2 e^{-\theta|q-q_0|} \right]^{1/2}.$$

Here, it is also taken into account that $\min_{p \in [0,1]} \phi'(p) \geq C$ (see assertion (iii) of Proposition 3.1); here the last constant is positive and depends only on ω_0 . Then the inequality for I gives that $I \leq C(\omega_0)|\mu_0|^{1/4}$. Combining this and (4.5), we arrive at the following inequality:

$$\int_S \left[\frac{H_p^3 f^2}{2} + \frac{(2h_p + H_p)w_p^2}{2h_p^2} \right] \frac{\phi'_0}{H_p^3} e^{-\theta|q-q_0|} dq dp \leq C(\omega_0)\sqrt{|\mu_0|}. \quad (4.6)$$

Furthermore, by virtue of the Schwarz inequality, we get that

$$\int_{q_0-1}^{q_0+1} \int_0^1 |w_p| dp dq \leq \left[\int_{q_0-1}^{q_0+1} \int_0^1 \frac{|w_p|^2}{h_p} dp dq \right]^{1/2} \left[\int_{q_0-1}^{q_0+1} \int_0^1 h_p dp dq \right]^{1/2},$$

and the last expression is bounded by $C(\omega_0)|\mu_0|^{1/4}$ in view of (4.6). This fact and the inequality $|w(q, 1)| \leq \int_0^1 |w_p(q, p)| dp$ give us that

$$\int_{q_0-1}^{q_0+1} |\eta(q) - d_+(r)| dq \leq C(\omega_0)|\mu_0|^{1/4} \quad \text{for all } q_0 \in \mathbb{R},$$

because $w(q, 1) = \eta(q) - d_+(r)$.

According to assertion (i) of Proposition 3.1, we have that $\mu_0(\lambda) \rightarrow 0$ as $\lambda \rightarrow \lambda_c$, and so the last inequality implies that for every $\epsilon > 0$ there exists $r' = r'(\epsilon, M, \omega_0)$ such that $|\eta(q) - d_+(r)| < \epsilon$ for all $q \in \mathbb{R}$ and all $r \in (r_c, r')$. Indeed, $|\eta'(q)| \leq M$ a.e. on \mathbb{R} being the second component of a solution of problem P_r^M .

Now we turn to assertion (b) for a solution of problem P_r^M . First, we show that ψ_y is separated from zero on ∂D when r is close to r_c .

Since the boundary conditions (1.3) and (1.4) imply that

$$\{1 + [\eta'(x)]^2\} \psi_y^2(x, \eta(x)) = 3r - 2\eta(x) \quad \text{for all } x \in \mathbb{R},$$

we have

$$[\psi_y(x, \eta(x))]^2 \geq [3r - 2\eta(x)]/(1 + M^2) \quad \text{for all } x \in \mathbb{R}.$$

Let us show that the expression on the right-hand side is separated from zero by a positive constant depending only on ω_0 . For this purpose we write

$$3r - 2\eta = 3r - 2d_+(r) + 2[\eta - d_+(r)] = [\lambda_+(r)]^2 - 2\Omega(1) + 2[\eta - d_+(r)],$$

where the last equality is a consequence of (2.3). In view of (2.5) we have

$$[\lambda_+(r)]^2 - \Omega(1) = \lambda_0^2 - \Omega(1) + [\lambda_+(r)]^2 - \lambda_0^2 \geq (\lambda_c - \lambda_0)\lambda_c/4,$$

but assertion (a) of Theorem 4.1 with $\epsilon = (\lambda_c - \lambda_0)\lambda_c/16$ yields that there exists $r'(M, \omega_0)$ such that

$$2|\eta - d_+(r)| \leq (\lambda_c - \lambda_0)\lambda_c/8 \quad \text{for all } r \in (r_c, r'].$$

Therefore,

$$[\psi_y(x, \eta(x))]^2 \geq (\lambda_c - \lambda_0)\lambda_c/[8(1 + M^2)] \quad \text{for all } x \in \mathbb{R}.$$

To estimate ψ_y near the bottom we use the problem (3.1)–(3.3) because the maximum principle (see [5], Theorem 1.4) is applicable to it. We choose λ_* so that $\sup_{x \in \mathbb{R}} \eta(x) = H(p; \lambda_*)$, and put $w(q, p) = h(q, p) - H(p; \lambda_*)$. Since $w(q, 0) = 0$ for all $q \in \mathbb{R}$ and $w(q, 1) < 0$, the maximum principle implies that $w < 0$ in S and $w_p(q, 0) < 0$. In other words, $h_p(q, 0) < H_p(0, \lambda_*)$, and so

$$\psi_y(x, 0) \geq [H_p(0, \lambda_*)]^{-1} \geq [H_p(0, \lambda_+(r))]^{-1} \geq \sqrt{(\lambda_c - \lambda_0)\lambda_c/4}.$$

Here the second inequality is a consequence of (3.4).

The obtained estimates of ψ_y on ∂D allow us to apply Theorem 8.26, [10], to the equation $\nabla^2 \psi_y + \omega'(\psi)\psi_y = 0$, thus demonstrating that ψ_y is separated from zero in a neighbourhood of ∂D . According to this theorem we have

$$\|\psi_y^-\|_{L^2(B_{6a}(Z))} \leq C(\omega_1) \inf_{B_{3a}(Z)} \psi_y^-, \quad (4.7)$$

where Z is an arbitrary point of ∂D ,

$$\psi_y^-(x, y) = \begin{cases} \min\{\psi_y, m\} & \text{when } (x, y) \in D, \\ m & \text{when } (x, y) \notin D \end{cases}$$

and $m = \inf_{B_{12a}(Z) \cap \partial D} \psi_y$. We recall that $B_a(Z)$ denotes the open circle of radius a centred at Z .

It is essential that the constant in (4.7) depends only on ω_1 and is independent of a . Indeed, Lemma 4.1 and inequality (2.5) imply that

$$C_1(\omega_0) < 3a < C_2(\omega_0),$$

where both constants are positive and depend only on ω_0 . Since the free surface is a Lipschitz curve with the constant $\sqrt{1 + M^2}$, a large part of the ball $B_{6a}(Z)$ belongs to the complement of D for every $Z \in \partial D$ (this is obvious for the flat bottom), and so $\|\psi_y^-\|_{L^2(B_{6a}(Z))} \geq C(M)m$. Combining this inequality and (4.7), we get that

$$\inf_{B_{3a}(Z) \cap D} \psi_y \geq C(M, \omega_1)m \quad \text{for every } Z \in \partial D.$$

The last expression is greater than a certain positive constant $C_1(M, \omega_1)$, and so the required inequality holds in the whole domain D because it is covered by circles $B_{3a}(Z)$ with $Z \in \partial D$. This completes the proof. \square

Now, we are going to estimate Hölder norms of derivatives for solutions of problem P_r^M . A bound for them will be obtained under the assumption that the horizontal component of the velocity field is uniformly separated from zero in \bar{D} .

Proposition 4.2. *Let $R \in (r_c, r_0)$ and $M > 0$. If problem P_r^M with $r \in (r_c, R]$ has a solution (ψ, η) , whose first component satisfies the inequality $\psi_y > \delta$ in \bar{D} with some $\delta > 0$, then there exist $\alpha \in (0, 1)$ and $C > 0$ (both depending only on M, R, δ and ω_1) such that the following inequalities hold:*

$$\|\psi\|_{C^{2,\alpha}(\bar{D})} \leq C \quad \text{and} \quad \|\eta\|_{C^{2,\alpha}(\mathbb{R})} \leq C. \quad (4.8)$$

Proof. The assumptions imposed on (ψ, η) and Proposition 4.1 imply that the corresponding solution h of problem (3.1)–(3.3) satisfies the following inequalities

$$0 < C(\omega_1, R) \leq h_p \leq \delta^{-1} \quad \text{in} \quad \bar{S} = \mathbb{R} \times [0, 1]. \quad (4.9)$$

Let us show that there exists $\alpha > 0$ such that the inequality

$$\|h\|_{C^{1,\alpha}(\bar{S})} \leq C'(R, \omega_1) \quad (4.10)$$

holds with a positive constant $C'(R, \omega_1)$.

Since ∂S consists of two straight lines and the Dirichlet boundary condition is fulfilled when $p = 0$, the only difficulty is to prove the estimate near the line $\{q \in \mathbb{R}, p = 1\}$ (cf. [10], Theorems 13.1 and 13.2). To overcome this difficulty we use the local estimates obtained in [20] (see the proof of Theorem 2.1, ch. 10) for the quasilinear equation (3.1) written in divergence form and complemented by the boundary condition (3.3); namely:

$$[a_1(h_q, h_p)]_q + [a_2(h_q, h_p)]_p - \omega = 0, \quad [a_2(h_q, h_p) + \phi(h)]_{p=1} = 0, \quad (4.11)$$

$$\text{where } a_1(h_q, h_p) = \frac{h_q}{h_p}, \quad a_2(h_q, h_p) = -\frac{1 + h_q^2}{2h_p^2} \text{ and } \phi(h) = 2h - 3r. \quad (4.12)$$

Indeed, all conditions on a_1, a_2 and ϕ required in the mentioned theorem are fulfilled in our case because the imposed conditions guarantee that the two-sided restriction (4.9) holds which, in its turn, yields (4.10).

The next step is to apply Theorem 11.2, [1], to the problem (4.11) and (4.12), which gives that $h \in C^{2,\alpha}(\bar{S})$, but does not provide a bound for the norm. To obtain such a bound we write the problem as follows:

$$\begin{aligned} h_{qq} - 2h_q h_p^{-1} h_{qp} + (1 + h_q^2) h_p^{-2} h_{pp} + \omega(p) h_p &= 0 \quad \text{in } S; \\ (1 + h_q^2) h_p^{-2} - (3r - 2h) &= 0 \quad \text{when } p = 1; \quad h = 0 \quad \text{when } p = 0. \end{aligned}$$

Now, we are in a position to use Theorem 3.1, [20], ch. 10, which gives a bound for $\|h\|_{C^{2,\alpha}(\bar{S})}$, depending only on M, R, δ and ω_1 . Therefore, in view of (3.4) and (4.9) both inequalities (4.8) are true. \square

5 Proof of Theorem 2.2

Let us prove the following assertion that is slightly stronger than Theorem 2.2.

Theorem 5.1. *There exist $r'' \in (r_c, r_0)$ and positive C and θ (r'' depends on M and ω_1 , whereas C and θ depend only on ω_0) such that any two solutions $(\psi^{(1)}, \eta^{(1)})$ and $(\psi^{(2)}, \eta^{(2)})$ of problem P_r^M with $r \in (r_c, r'']$ satisfy the inequality*

$$\begin{aligned} \int_S |\nabla h^{(1)} - \nabla h^{(2)}|^2 e^{-\theta|x-x_0|} dq dp \\ \leq C \left[|\eta^{(1)}(q_0) - \eta^{(2)}(q_0)|^2 + |\eta_x^{(1)}(q_0) - \eta_x^{(2)}(q_0)|^2 \right]. \end{aligned} \quad (5.1)$$

Here $q_0 \in \mathbb{R}$ is arbitrary, whereas $h^{(1)}$ and $h^{(2)}$ correspond to $(\psi^{(1)}, \eta^{(1)})$ and $(\psi^{(2)}, \eta^{(2)})$ through the partial hodograph transform.

It is easy to see that Theorem 2.2 is a consequence of Theorem 5.1. Our proof of the latter theorem is based on another form of problem (3.1)–(3.3); namely, a system of Hamilton's equations proposed in [9] (see (5.2) and (5.3) below).

5.1 A System of First Order Equivalent to Problem (3.1)–(3.3)

Let us write a system equivalent to the problem (3.1)–(3.3), for which purpose we complement h by the unknown function $f = h_q/h_p$, thus obtaining for $(q, p) \in S$:

$$\begin{cases} h_q = fh_p, \\ f_q = -\frac{1}{2}\mathcal{A}(f, h) + \omega(p). \end{cases} \quad (5.2)$$

$$(5.3)$$

Here \mathcal{A} is a nonlinear operator which is convenient to define by virtue of the integral identity

$$\int_0^1 \mathcal{A}(f, h) \Phi dp = \int_0^1 (f^2 + h_p^{-2}) \Phi_p dp - [\{3r - 2h(q, p)\} \Phi(p)]_{p=1},$$

which must hold for every $\Phi \in H_0^1(0, 1)$; this space consists of continuous functions that vanish at $p = 0$ and have derivatives in $L^2(0, 1)$. The last identity is well defined because the functions f and h_p^{-1} are bounded when (ψ, η) is a solution of problem P_r^M (see Proposition 4.1).

This weak formulation allows us to use the technique developed in [18] (see also [19]) for ordinary differential equations with operator coefficients. It is worth mentioning that the explicit form of the operator \mathcal{A} (the differential expression and boundary operators) can be found in [9].

5.1.1 Linearization Near a Stream Solution

Let us linearize the equations (5.2) and (5.3) near the stream solution $H(p, \lambda_+(r))$. For this purpose we put $h = H + w$, thus obtaining that the pair (w, f) must satisfy the system

$$\begin{cases} w_q - fH_p = \mathcal{N}_1(w, f), \\ f_q - \mathcal{L}(w) = \mathcal{N}_2(w, f). \end{cases} \quad (5.4)$$

$$(5.5)$$

Here $(q, p) \in S$, $\mathcal{N}_1(w, f) = fw_p$, whereas the operators \mathcal{L} and \mathcal{N}_2 are defined in the same way as \mathcal{A} . Their action on functions given on the cross-section of S is described by the following integral identities:

$$\begin{aligned} \int_0^1 \mathcal{L}(w)\Phi \, dp &= \int_0^1 \frac{w_p \Phi_p}{H_p^3} \, dp - [w\Phi]_{p=1}, \\ \int_0^1 \mathcal{N}_2(w)\Phi \, dp &= - \int_0^1 \left[\frac{w_p^3}{H_p^3 h_p^2} + \frac{3}{2} \frac{w_p^2}{H_p^2 h_p^2} + \frac{f^2}{2} \right] \Phi_p \, dp. \end{aligned}$$

They must be fulfilled for every $\Phi \in H_0^1(0, 1)$. Note that \mathcal{L} is nothing else than the operator of the spectral problem (3.6)–(3.8) represented in a weak form.

Using Theorem 4.1 and Proposition 4.2, we prove the following assertion.

Lemma 5.1. *For any $\epsilon, M > 0$ there exists $r^* \in (r_c, r_0)$ depending only on M, ϵ and ω_1 such that if (ψ, η) is a solution of problem P_r^M with $r \in (r_c, r^*]$, then the following inequality holds:*

$$\|w\|_{C^2(\bar{S})} + \|f\|_{C^1(\bar{S})} < \epsilon.$$

Proof. Let the pair (w, f) corresponds to a solution (ψ, η) of problem P_r^M with $r \in (r_c, r']$. Here $r' = r'(\epsilon_*, M, \omega_1) \in (r_c, r_0)$ that exists for every $\epsilon_* > 0$ (it will be chosen later depending on the values of ϵ, M and ω_1) by assertion (a) of Theorem 4.1. Moreover, assertion (b) of Theorem 4.1 guarantees that there exist $r'' = r''(M, \omega_1)$ and $\delta_*(M, \omega_1) > 0$ such that both assertions of that theorem are true for any solution (ψ, η) of problem P_r^M with $r \in (r_c, \min\{r', r''\}]$. Without loss of generality we suppose that condition (2.5) is also fulfilled for r' and r'' .

According to Proposition 4.2, we have that $\psi \in C^{2,\alpha}(\bar{D})$ for some $\alpha \in (0, 1)$. For both functions ψ and η (the latter bounds D from above) their $C^{2,\alpha}$ -norms are bounded by the same constant depending on M and ω_1 , but independent of ϵ_* ; that is,

$$\|\psi\|_{C^{2,\alpha}(\bar{D})}, \|\eta\|_{C^{2,\alpha}(\mathbb{R})} \leq C(M, \omega_1). \quad (5.6)$$

We recall that $w = h - H$ and $f = h_q/h_p$, where $H = H(p; \lambda_+(r))$ is a stream solution used in the proof of Theorem 4.1 and h corresponds to (ψ, η) through the partial hodograph transform. Since $w(q, 1) = \eta(q) - d_+(r)$, assertion (a) of Theorem 4.1 gives that $|w(q, 1)| < \epsilon_*$ for all $q \in \mathbb{R}$. Moreover, we have

that $w(q, 0) = 0$, and so the maximum principle yields that $\|w\|_{L^\infty(\overline{S})} < \epsilon_*$. Combining this and (5.6), we conclude by virtue of interpolation argument that

$$\|w\|_{C^2(\overline{S})}, \|f\|_{C^1(\overline{S})} < \epsilon_*^\gamma C(M, \omega_1), \quad i = 1, 2,$$

where $\gamma = \frac{\alpha}{2(1+\alpha)}$. Now, choosing ϵ_* so that $\epsilon_*^\gamma C(M, \omega_1) = \epsilon$, we complete the proof. \square

5.1.2 Spectral Splitting

The spectral problem related to the system (5.4), (5.5) is as follows:

$$\begin{cases} f H_p = \sigma w, \\ \mathcal{L}(w) = \sigma f, \end{cases} \quad (5.7)$$

$$(5.8)$$

where $p \in (0, 1)$. It is clear that $(f, w) \in L^2(0, 1) \times H_0^1(0, 1)$ is an eigensolution corresponding to σ , if and only if $w = \phi$ and $f = \sigma \phi H_p^{-1}$, where ϕ is an eigenfunction of the problem (3.6)–(3.8) corresponding to the eigenvalue $\mu = \sigma^2$. Hence the spectrum of (5.7)–(5.8) is the sequence $\{\sigma_i\}_{i=0}^\infty$ with $\sigma_i = \sqrt{\mu_i}$, where σ_i is real and positive for $i \geq 1$. The corresponding eigensolutions are $(w_i, f_i) = (\phi_i, \sigma_i \phi_i H_p^{-1})$. Since μ_0 is negative, there are two complex eigenvalues $\sigma_0^\pm = \pm i \sqrt{|\mu_0|}$ and the corresponding two-dimensional eigenspace is spanned by $(\phi_0, 0)$ and $(0, \phi_0 H_p^{-1})$.

Given a fixed $q \in \mathbb{R}$ and the real-valued functions $w(q, p)$ and $f(q, p)$, then there are the following spectral decompositions:

$$w(q, p) = \sum_{i=0}^\infty w_i(q) \phi_i(p), \quad f(q, p) = \sum_{i=0}^\infty f_i(q) \phi_i(p) H_p^{-1}.$$

In view of these formulae we define two projectors $\mathcal{P}(w, f) = (\mathcal{P}_1 w, \mathcal{P}_2 f)$ and $\mathcal{Q}(w, f) = (w, f) - \mathcal{P}(w, f)$, where

$$\mathcal{P}_1(w) = \phi_0 \int_0^1 w \phi_0 H_p^{-1} dp \quad \text{and} \quad \mathcal{P}_2(f) = \phi_0 H_p^{-1} \int_0^1 f \phi_0 dp. \quad (5.9)$$

This leads to the following spectral splitting:

$$(w, f) = \mathcal{P}(w, f) + \mathcal{Q}(w, f).$$

Here the first term is equal to $(w_0 \phi_0, f_0 \phi_0 H_p^{-1})$, whereas the second one we denote by $(\tilde{w}, \tilde{\zeta})$. Applying the projectors \mathcal{Q} and \mathcal{P} to the equations (5.4) and (5.5), we obtain

$$\begin{cases} \tilde{w}_q = \tilde{\zeta} H_p + (I - \mathcal{P}_1) \mathcal{N}_1(w, f), \\ \tilde{\zeta}_q = \mathcal{L}(\tilde{w}) + (I - \mathcal{P}_2) \mathcal{N}_2(w, f) \end{cases} \quad (5.10)$$

$$(5.11)$$

for $(q, p) \in S$, and

$$\begin{cases} (w_0)_q = f_0 + \int_0^1 \mathcal{N}_1(w, f) \phi_0 H_p^{-1} dp, \end{cases} \quad (5.12)$$

$$\begin{cases} (f_0)_q = \mu_0 w_0 + \int_0^1 \mathcal{N}_2(w, f) \phi_0 dp \end{cases} \quad (5.13)$$

for $q \in \mathbb{R}$. Let us prove that \mathcal{L} is a positive operator.

Lemma 5.2. *Let r satisfy condition (2.5). For all $w \in H_0^1(0, 1)$ orthogonal to the function $\phi_0 H_p^{-1}$ in $L^2(0, 1)$ the inequality*

$$\int_0^1 \mathcal{L}w \cdot w dp \geq C \|w\|_{H^1(0,1)}^2$$

holds with a positive constant C depending only on ω_0 .

Proof. Using the spectral representation of w , one immediately obtains

$$\int_0^1 \mathcal{L}w \cdot w dp \geq \mu_1 \int_0^1 |w|^2 H_p^{-1} dp, \quad (5.14)$$

where $\mu_1 > 0$ satisfies assertion (ii) of Proposition 3.1. On the other hand, the definition of \mathcal{L} yields that

$$\int_0^1 \mathcal{L}w \cdot w dp = \int_0^1 \frac{w_p^2}{H_p^3} dp - w^2(1).$$

Using the Schwarz inequality, we estimate $w^2(1) = \int_0^1 w w_p dp / 2$ by

$$\frac{1}{2} \left[\int_0^1 w^2 H_p^3 dp \right]^{1/2} \left[\int_0^1 w_p^2 H_p^{-3} dp \right]^{1/2} \leq \frac{\mathfrak{M}^4}{2} \int_0^1 w^2 H_p^{-1} dp + \frac{1}{2} \int_0^1 w_p^2 H_p^{-3} dp,$$

where $\mathfrak{M} = \max_{[0,1]} H_p$. Then we obtain that

$$\int_0^1 \mathcal{L}w \cdot w dp \geq \frac{1}{2} \int_0^1 \frac{w_p^2}{H_p^3} dp - \frac{\mathfrak{M}^4}{2} \int_0^1 \frac{w^2}{H_p} dp.$$

Combining this inequality and (5.14), we arrive at the required inequality. \square

5.1.3 Estimates for the Linearized Problem

Let us consider the linearized system (5.10)–(5.13). The linear part of (5.10) and (5.11) is as follows:

$$\begin{cases} \tilde{\xi}_q - \tilde{\zeta} H_p = g^{(1)}, \end{cases} \quad (5.15)$$

$$\begin{cases} \tilde{\zeta}_q - \mathcal{L}(\tilde{\xi}) = g^{(2)}, \end{cases} \quad (5.16)$$

where $(q, p) \in S$. In our considerations, it is sufficient to regard $g^{(1)}$ and $g^{(2)}$ as functions belonging to $C(\mathbb{R}; L^2(0, 1))$, whereas $\tilde{\xi}$ and $\tilde{\zeta}$ are from $C^1(\mathbb{R}; H_0^1(0, 1))$ and $C^1(\mathbb{R}; L^2(0, 1))$, respectively. Moreover, for every $q \in \mathbb{R}$ the functions $\tilde{\xi}(q, \cdot)$ and $\tilde{\zeta}(q, \cdot)$ are orthogonal in $L^2(0, 1)$ to $\phi_0 H_p^{-1}$ and ϕ_0 , respectively. Furthermore, the linearized equations (5.12) and (5.13) are as follows:

$$\begin{cases} \xi'_0 - \zeta_0 = g_0^{(1)}, \\ \zeta'_0 + |\mu_0| \xi_0 = g_0^{(2)}. \end{cases} \quad (5.17)$$

$$(5.18)$$

Here $q \in \mathbb{R}$, $\xi_0, \zeta_0 \in C^1(\mathbb{R})$, $g_0^{(1)}, g_0^{(2)} \in C(\mathbb{R})$ and $'$ denotes d/dq .

Our aim is to estimate solutions of these systems using the following norm:

$$\|w\|_{\theta, q_0}^2 = \int_S |w|^2 e^{-\theta|q-q_0|} dq dp, \quad \text{where } q_0 \in \mathbb{R} \text{ and } \theta > 0. \quad (5.19)$$

Lemma 5.3. *Let $(\tilde{\xi}, \tilde{\zeta})$ be a solution of the system (5.15), (5.16). If $\tilde{\xi}$ and $\tilde{\xi}_q$ belong to $L^\infty(\mathbb{R}; H_0^1(0, 1))$ and $\tilde{\zeta}$ and $\tilde{\zeta}_q$ belong to $L^\infty(\mathbb{R}; L^2(0, 1))$, then there exists $\theta_0 \in (0, 1/4)$ depending on ω_0 such that for any $\theta \in (0, \theta_0]$ and $q_0 \in \mathbb{R}$ the following inequality holds:*

$$\|\tilde{\zeta}\|_{\theta, q_0}^2 + \|\tilde{\xi}\|_{\theta, q_0}^2 + \|\tilde{\xi}_p\|_{\theta, q_0}^2 \leq C \left| \int_S [g^{(1)} \tilde{\zeta} + g^{(2)} \tilde{\xi}] e^{-\theta|q-q_0|} dq dp \right|, \quad (5.20)$$

Here the positive constant C depends only on ω_0 .

Unlike usual estimates, in which solution's norm is estimated by a norm of the term on the right-hand side, the inequality (5.20) has an expression involving g_1 and g_2 on the right-hand side. The reason for this is the fact that g_1 and g_2 are some integral functionals (see (5.10) and (5.11)).

Proof. Let us multiply (5.15) and (5.16) by $\tilde{\zeta}$ and $\tilde{\xi}$, respectively, and integrate the results over $(0, 1)$. Summing up, we obtain that

$$\int_0^1 [\tilde{\xi} \tilde{\zeta}]_q dp = \int_0^1 [\tilde{\zeta}^2 H_p + (\mathcal{L} \tilde{\xi}) \tilde{\zeta}] dp + \int_0^1 [g^{(1)} \tilde{\zeta} + g^{(2)} \tilde{\xi}] dp = I_1 + I_2.$$

Applying Lemma 5.2, we estimate I_1 from below as follows:

$$I_1 \geq \mathfrak{m} \|\tilde{\zeta}(q, \cdot)\|_{L^2(0, 1)}^2 + C(\omega_0) \|\tilde{\xi}(q, \cdot)\|_{H^1(0, 1)}^2,$$

where C is the constant from Lemma 5.2. Let $\theta_0 = \min\{C(\omega_0), \mathfrak{m}\}/2$, multiplying the last inequality by $e^{-\theta|q-q_0|}$ with $\theta \in (0, \theta_0)$ and $q_0 \in \mathbb{R}$, we obtain after integration over \mathbb{R}

$$\|\tilde{\zeta}\|_{\theta, q_0}^2 + \|\tilde{\xi}\|_{\theta, q_0}^2 + \|\tilde{\xi}_p\|_{\theta, q_0}^2 \leq \frac{1}{2\theta_0} \left[|I_2| + \left| \int_S (\tilde{\xi} \tilde{\zeta})_q e^{-\theta|q-q_0|} dq dp \right| \right].$$

Integrating by parts in the last integral, we see that its absolute value is less than or equal to $\theta_0 \left(\|\tilde{\zeta}\|_{\theta, q_0}^2 + \|\tilde{\xi}\|_{\theta, q_0}^2 \right)$, which leads to the following inequality:

$$\|\tilde{\zeta}\|_{\theta, q_0}^2 + \|\tilde{\xi}\|_{\theta, q_0}^2 + \|\tilde{\xi}_p\|_{\theta, q_0}^2 \leq \frac{1}{\theta_0} \left| \int_S \left[g^{(1)} \tilde{\zeta} + g^{(2)} \tilde{\xi} \right] e^{-\theta|q-q_0|} dq dp \right|.$$

The proof is complete. \square

Lemma 5.4. *Let (ξ_0, ζ_0) be a bounded solution of the system (5.17), (5.18) such that ξ'_0 and ζ'_0 are also bounded. Then the inequality*

$$\|\zeta_0\|_{\theta, q_0}^2 \leq C_1(\omega_0) \left[\|\xi_0\|_{\theta, q_0}^2 + \|g_0^{(1)}\|_{\theta, q_0}^2 + \|g_0^{(2)}\|_{\theta, q_0}^2 \right] \quad (5.21)$$

holds for all $\theta \in (0, 1/4]$, $q_0 \in \mathbb{R}$; the positive constant C_1 depends only on ω_0 .

Moreover, if $\xi_0(q_0) = \zeta_0(q_0) = 0$ for some $q_0 \in \mathbb{R}$, then

$$\|\xi_0\|_{\theta, q_0}^2 + \|\zeta_0\|_{\theta, q_0}^2 \leq C_2(\omega_0, \theta) \|g_0^{(1)}\|_{\theta, q_0}^2 + C_3(\theta) \|g_0^{(2)}\|_{\theta, q_0}^2 \quad (5.22)$$

holds with an arbitrary $\theta > 0$; the positive constant C_2 depends on θ and ω_0 , while C_3 depends only on θ .

Proof. Let us prove (5.21) first. Multiplying the equations (5.17) and (5.18) by ζ_0 and ξ_0 , respectively, and summing up, we obtain

$$(\xi_0 \zeta_0)' - \zeta_0^2 + |\mu_0| \xi_0^2 = g_0^{(1)} \zeta_0 + g_0^{(2)} \xi_0.$$

This gives that

$$\frac{3}{4} \zeta_0^2 \leq (\xi_0 \zeta_0)' + 4 \left[g_0^{(1)} \right]^2 + \left[g_0^{(2)} \right]^2 + (1 + |\mu_0|) \xi_0^2.$$

Let us multiply this inequality by $e^{-\theta|q-q_0|}$ and integrate over \mathbb{R} . This yields

$$\frac{3}{4} \|\zeta_0\|_{\theta, q_0}^2 \leq \int_{\mathbb{R}} (\xi_0 \zeta_0)' e^{-\theta|q-q_0|} dq + C \left[\|g_0^{(1)}\|_{\theta, q_0} + \|g_0^{(2)}\|_{\theta, q_0} + \|\xi\|_{\theta, q_0} \right].$$

Estimating the integral on the right-hand side

$$\int_{\mathbb{R}} (\xi_0 \zeta_0)' e^{-\theta|q-q_0|} dq \leq \theta \left(\|\zeta_0\|_{\theta, q_0}^2 + \|\xi_0\|_{\theta, q_0}^2 \right),$$

we see that for $\theta \leq \theta_0 = 1/4$ the required inequality follows from the last two estimates.

Let turn to the second assertion. We multiply (5.18) by ζ_0 and take into account (5.17), thus obtaining

$$\zeta_0 \zeta'_0 + |\mu_0| \xi_0 \xi'_0 = |\mu_0| \xi_0 g_0^{(1)} + \zeta_0 g_0^{(2)}.$$

Multiplying this by $e^{-\theta|q-q_0|}$, we integrate the result over $(q_0, +\infty)$ which gives

$$\theta \int_{q_0}^{+\infty} [\zeta_0^2 + |\mu_0| \xi_0^2] e^{-\theta|q-q_0|} dq = \int_{q_0}^{+\infty} [|\mu_0| \xi_0 g_0^{(1)} + \zeta_0 g_0^{(2)}] e^{-\theta|q-q_0|} dq$$

after integration by parts. Applying the Schwarz inequality to the right-hand side, we arrive at

$$\frac{\theta}{2} \int_{q_0}^{+\infty} [\zeta_0^2 + |\mu_0| \xi_0^2] e^{-\theta|q-q_0|} dq \leq \frac{2|\mu_0|}{\theta} \|g_0^{(1)}\|_{\theta, q_0}^2 + \frac{2}{\theta} \|g_0^{(2)}\|_{\theta, q_0}^2. \quad (5.23)$$

Now, we multiply (5.17) by $\xi_0 e^{-\theta|q-q_0|}$ and integrate the result over $(q_0, +\infty)$. This gives after integration by parts the following inequality:

$$\theta \int_{q_0}^{+\infty} \xi_0^2 e^{-\theta|q-q_0|} dq = \int_{q_0}^{+\infty} [\zeta_0 \xi_0 + \xi_0 g_0^{(1)}] e^{-\theta|q-q_0|} dq.$$

Applying the Schwarz inequality to the right-hand side, we obtain

$$\frac{\theta}{2} \int_{q_0}^{+\infty} \xi_0^2 e^{-\theta|q-q_0|} dq \leq \frac{4}{\theta} \left[1 + \frac{2|\mu_0|}{\theta} \right] \|g_0^{(1)}\|_{\theta, q_0}^2 + \frac{8}{\theta^2} \|g_0^{(2)}\|_{\theta, q_0}^2,$$

where (5.23) is also taken into account. Combining the last inequality and (5.23), we arrive at

$$\int_{q_0}^{+\infty} [\xi_0^2 + \zeta_0^2] e^{-\theta|q-q_0|} dq \leq C_1(\omega_0, \theta) \|g_0^{(1)}\|_{\theta, q_0}^2 + C_2(\theta) \|g_0^{(2)}\|_{\theta, q_0}^2.$$

In the same way we estimate $\int_{-\infty}^{q_0} [\xi_0^2 + \zeta_0^2] e^{-\theta|q-q_0|} dq$, which completes the proof. \square

5.2 Proof of Theorem 5.1

Let $(\psi^{(1)}, \eta^{(1)})$ and $(\psi^{(2)}, \eta^{(2)})$ be two arbitrary solutions of problem P_r^M ; here M is a positive number and the Bernoulli constant r satisfies condition (2.5).

For the functions $h^{(1)}$ and $h^{(2)}$ that correspond to $(\psi^{(1)}, \eta^{(1)})$ and $(\psi^{(2)}, \eta^{(2)})$, respectively, through the partial hodograph transform, we put

$$w^{(i)} = h^{(i)} - H \quad \text{and} \quad f^{(i)} = h_q^{(i)} / h_p^{(i)}, \quad i = 1, 2,$$

where $H = H(p; \lambda_+(r))$. Let

$$\xi = w^{(1)} - w^{(2)} \quad \text{and} \quad \zeta = f^{(1)} - f^{(2)},$$

for which we consider the following spectral splitting (see Section 5.1.2 for the corresponding notation):

$$\xi = \xi_0 \phi_0 + \tilde{\xi} \quad \text{and} \quad \zeta = \zeta_0 \phi_0 H_p^{-1} + \tilde{\zeta}.$$

Here the functions $\tilde{\xi}H_p^{-1}$ and $\tilde{\zeta}$ are orthogonal to ϕ_0 in $L^2(0, 1)$.

Let us outline our proof of Theorem 5.1. First, we estimate the θ, q_0 -norm (see (5.19) for its definition) of $\tilde{\xi}$ and $\tilde{\zeta}$ by the same norm of the one-dimensional projections ξ_0 and ζ_0 (see Lemma 5.5 below). Next we estimate the norm of ξ_0 by its Cauchy data (see Lemma 5.6). The last step is to estimate the Cauchy data of ξ_0 at some point by the Cauchy data of $\eta^{(1)} - \eta^{(2)}$ at the same point (see Lemma 5.7 below).

5.2.1 Lemmas

Assuming that both solutions $(w^{(i)}, f^{(i)})$, $i = 1, 2$ are small, we estimate the difference of their projections by the norm of the function ξ_0 only.

Lemma 5.5. *Let r satisfy (2.5). Then there exist θ_0, ϵ_0 and $C(\omega_0)$ (all positive) such that the inequality*

$$\|\tilde{\zeta}\|_{\theta, q_0}^2 + \|\tilde{\xi}\|_{\theta, q_0}^2 + \|\tilde{\xi}_p\|_{\theta, q_0}^2 \leq \epsilon C(\omega_0) \|\xi_0\|_{\theta, q_0}^2$$

holds for all $\theta \in (0, \theta_0]$, $q_0 \in \mathbb{R}$ and $\epsilon \in (0, \epsilon_0]$ provided

$$\epsilon = \max_{i=1,2} \left(\|w^{(i)}\|_{C^2(\overline{S})} + \|f^{(i)}\|_{C^1(\overline{S})} \right) \leq \epsilon_0.$$

The constants ϵ_0 and θ_0 depend only on ω_0 and the second of them is the same as in Lemma 5.3, and so allows us to apply Lemma 5.4.

Proof. Since $\tilde{\xi}$ and $\tilde{\zeta}$ solve the system (5.15), (5.16) with the following right-hand side terms (see (5.9) for the definition of \mathcal{P}_i):

$$g^{(i)} = (I - \mathcal{P}_i) \left[\mathcal{N}_i \left(w^{(1)}, f^{(1)} \right) - \mathcal{N}_i \left(w^{(2)}, f^{(2)} \right) \right], \quad i = 1, 2,$$

Lemma 5.3 implies that there exists $\theta_0 > 0$ such that the inequality (5.20) holds for $\tilde{\xi}$ and $\tilde{\zeta}$ and all $\theta \in (0, \theta_0]$. Hence it remains to estimate the term on the right-hand side of (5.20); its integrand consists of two terms that have $g^{(1)}$ and $g^{(2)}$ as factors.

The Schwarz inequality applied to the first term gives

$$\left| \int_0^1 g^{(1)} \tilde{\zeta} \, dp \right| \leq I \left[\int_0^1 \left\{ g^{(1)} \right\}^2 \, dp \right]^{\frac{1}{2}}, \quad \text{where } [I(q)]^2 = \int_0^1 \left(\tilde{\xi}^2 + \tilde{\xi}_p^2 + \tilde{\zeta}^2 \right) \, dp.$$

Inequality (2.5) together with Proposition 3.1 yield that the operator \mathcal{P}_1 is bounded in $L^2(0, 1)$ and its norm does not exceed a constant depending only on ω_0 . Thus, we have

$$\int_0^1 \left[g^{(1)} \right]^2 \, dp \leq C(\omega_0) \int_0^1 \left[\mathcal{N}_1 \left(w^{(1)}, f^{(1)} \right) - \mathcal{N}_1 \left(w^{(2)}, f^{(2)} \right) \right] \, dp,$$

whereas the definition of \mathcal{N}_1 gives

$$\begin{aligned} \left| \mathcal{N}_1 \left(w^{(1)}, f^{(1)} \right) - \mathcal{N}_1 \left(w^{(2)}, f^{(2)} \right) \right| &= \left| w^{(1)} f^{(1)} - w^{(2)} f^{(2)} \right| \\ &\leq \epsilon C(\omega_0)(I + I_0), \quad \text{where } [I_0(q)]^2 = \xi_0^2 + \zeta_0^2. \end{aligned}$$

Therefore,

$$\left| \int_0^1 g^{(1)} \tilde{\zeta} \, dp \right| \leq \epsilon C'(\omega_0)(I^2 + I_0^2).$$

Let us turn to estimating the second term on the right-hand side of (5.20):

$$\int_0^1 g^{(2)} \tilde{\xi} \, dp = \int_0^1 \tilde{\xi} (I - \mathcal{P}_2) \left[\mathcal{N}_2 \left(w^{(1)}, f^{(1)} \right) - \mathcal{N}_2 \left(w^{(2)}, f^{(2)} \right) \right] \, dp = J_1 - J_2.$$

Here J_1 stands for the integral whose integrand is the product of the square bracket and $\tilde{\xi}$, whereas the operator \mathcal{P}_2 is applied to the same square bracket in J_2 . Without loss of generality, we can assume that $\epsilon \leq 1$. Then the definition of \mathcal{N}_2 gives that

$$|J_1| \leq \epsilon C(\omega_0)(I^2 + I_0^2).$$

On the other hand, we have

$$|J_2| \leq C(\omega_0) \int_0^1 |\tilde{\xi}| \, dp \cdot \left| \int_0^1 \left[\mathcal{N}_2 \left(w^{(1)}, f^{(1)} \right) - \mathcal{N}_2 \left(w^{(2)}, f^{(2)} \right) \right] \phi_0 \, dp \right|.$$

Using the definition of \mathcal{N}_2 , we conclude that the last absolute value is less than or equal to $\epsilon C(\omega_0)(I + I_0)$. Combining this fact and the previous estimate, we get that

$$|J_2| \leq \epsilon C(\omega_0)(I^2 + I_0^2).$$

Let $\theta \in (0, \theta_0]$, where θ_0 is the same as in Lemma 5.3. Then inequality (5.20) and the estimates obtained above imply that

$$\begin{aligned} \int_{-\infty}^{\infty} [I(q)]^2 e^{-\theta|q-q_0|} \, dq &\leq \epsilon C(\omega_0) \int_{-\infty}^{\infty} [I(q)]^2 e^{-\theta|q-q_0|} \, dq \\ &\quad + \epsilon C(\omega_0) \int_{-\infty}^{\infty} [I_0(q)]^2 e^{-\theta|q-q_0|} \, dq. \end{aligned} \tag{5.24}$$

In order to estimate the last term on the right-hand side, we apply inequality (5.21), thus obtaining

$$\int_{-\infty}^{\infty} [I_0(q)]^2 e^{-\theta|q-q_0|} \, dq \leq C_1(\omega_0) \left[\|\xi_0\|_{\theta, q_0}^2 + \|g_0^{(1)}\|_{\theta, q_0}^2 + \|g_0^{(2)}\|_{\theta, q_0}^2 \right],$$

where

$$g_0^{(i)} = \int_0^1 \left[\mathcal{N}_i \left(w^{(1)}, f^{(1)} \right) - \mathcal{N}_i \left(w^{(2)}, f^{(2)} \right) \right] \phi_0 [H_p^{-1}]^{|i-1|} \, dp, \quad i = 1, 2.$$

As above, one derives the following estimate:

$$\|g_0^{(1)}\|_{\theta, q_0}^2 + \|g_0^{(2)}\|_{\theta, q_0}^2 \leq \epsilon C'(\omega_0)(I^2 + I_0^2).$$

Therefore, if $\epsilon \leq [C_1 C']^{-1}/2$, we see that $\|I_0\|_{\theta, q_0}^2 \leq 2C_1(\omega_0)\|\xi_0\|_{\theta, q_0}^2 + I^2$. Combining this inequality and (5.24), we complete the proof by taking ϵ less than or equal to $\epsilon_0 = \min\{[C_1 C']^{-1}/2, C^{-1}/2\}$. \square

In the next lemma, we estimate the θ, q_0 -norm of ξ_0 by the Cauchy data of this function at q_0 .

Lemma 5.6. *Let r satisfy (2.5). Then there exist $\theta_0 > 0$ (that from Lemma 5.5 can be used) and $\epsilon_0 > 0$ (both depend only on ω_0) such that the inequality*

$$\|\xi_0\|_{\theta, q_0}^2 \leq C [\xi_0^2(q_0) + \xi_0'^2(q_0)] \quad (5.25)$$

holds for all $\theta \in (0, \theta_0]$ and $q_0 \in \mathbb{R}$ provided

$$\max_{i=1,2} \left\{ \|w^{(i)}\|_{C^2(\overline{\mathbb{S}})} + \|f^{(i)}\|_{C^1(\overline{\mathbb{S}})} \right\} \leq \epsilon_0.$$

The positive constant C depends only on θ and ω_0 .

Proof. It follows from (5.12) and (5.13) that

$$\xi_0'' + |\mu_0|\xi_0 = g, \quad q \in \mathbb{R}, \quad (5.26)$$

where

$$\begin{aligned} g &= \int_0^1 \left[\mathcal{N}_2 \left(w^{(1)}, f^{(1)} \right) - \mathcal{N}_2 \left(w^{(2)}, f^{(2)} \right) \right] \phi_0 \, dp \\ &\quad - \frac{\partial}{\partial q} \int_0^1 \left[\mathcal{N}_1 \left(w^{(1)}, f^{(1)} \right) - \mathcal{N}_1 \left(w^{(2)}, f^{(2)} \right) \right] \phi_0 H_p^{-1} \, dp. \end{aligned}$$

Then the solution ξ_0 of (5.26) has the following form:

$$\begin{aligned} \xi_0(q) &= \xi_0(q_0) \cos \sqrt{|\mu_0|}(q - q_0) + \frac{\xi_0'(q_0)}{\sqrt{|\mu_0|}} \sin \sqrt{|\mu_0|}(q - q_0) \\ &\quad + \frac{1}{|\sqrt{\mu_0}|} \int_{q_0}^q g(q') \sin \sqrt{|\mu_0|}(q - q') \, dq', \quad q_0 \in \mathbb{R}. \end{aligned}$$

In order to prove (5.25) it is sufficient to estimate the θ, q_0 -norm of the right-hand side in the last equality. The norm of the first two terms is bounded by $C(\theta, \omega_0) \sqrt{\xi_0^2(q_0) + \xi_0'^2(q_0)}$. Turning to the last term, we see that it is sufficient to prove the inequality

$$\frac{1}{|\sqrt{\mu_0}|} \left\| \int_{q_0}^q g(q') \sin \sqrt{|\mu_0|}(q - q') \, dq' \right\|_{\theta, q_0} \leq \frac{1}{2} \|\xi_0\|_{\theta, q_0}. \quad (5.27)$$

Let us apply Lemma 5.4 (inequality (5.22)) to the function

$$\frac{1}{|\sqrt{\mu_0}|} \int_{q_0}^q g(q') \sin \sqrt{|\mu_0|}(q - q') \, dq'.$$

It is clear that this function satisfies the same equation (5.26) as ξ_0 , but with the homogeneous initial conditions at q_0 . Therefore, Lemma 5.4 applied to this function and its derivative instead of ξ_0 and ζ_0 , respectively, gives

$$\frac{1}{\mu_0} \left\| \int_{q_0}^q g(q') \sin \sqrt{|\mu_0|}(q - q') \, dq' \right\|_{\theta, q_0}^2 \leq C(\omega_0, \theta) \|g\|_{\theta, q_0}^2.$$

The last inequality is valid for all $\theta > 0$. Furthermore, according to the definition of g , Lemma 5.5 shows that the inequality

$$\|g\|_{\theta, q_0}^2 \leq \epsilon C'(\omega_0, \theta) \|\xi_0\|_{\theta, q_0}^2$$

holds for all $\theta \in (0, \theta_0]$ provided $\epsilon \in (0, \epsilon'_0]$ (here θ_0 and ϵ'_0 are the constants from Lemma 5.5), for which purpose we take

$$\epsilon = \max_{i=1,2} \left\{ \|w^{(i)}\|_{C^2(\overline{S})} + \|f^{(i)}\|_{C^1(\overline{S})} \right\}.$$

Finally, combining the last two inequalities, we arrive at (5.27), provided

$$\epsilon \leq \epsilon_0 = \min\{\epsilon'_0, [CC']^{-1}/2\},$$

and this completes the proof. \square

Now we estimate the Cauchy data of ξ_0 by the Cauchy data of ξ at the same point.

Lemma 5.7. *Let r satisfy (2.5). Then there exist positive constants C and ϵ_0 depending only on ω_0 such that if $\max_{i=1,2} \left\{ \|w^{(i)}\|_{C^2(\overline{S})} + \|f^{(i)}\|_{C^1(\overline{S})} \right\} \leq \epsilon_0$, then the inequality*

$$\xi_0^2(q_0) + \xi_0'^2(q_0) \leq C(\omega_0) [\xi^2(q_0, 1) + \xi_q^2(q_0, 1)]$$

holds for all $q_0 \in \mathbb{R}$.

Proof. It follows from the spectral splitting that

$$\xi_0(q_0) = \phi_0^{-1}(1)[\xi(q_0, 1) - \tilde{\xi}(q_0, 1)], \quad \xi_0'(q_0) = \phi_0^{-1}(1)[\xi_q(q_0, 1) - \tilde{\xi}_q(q_0, 1)]. \quad (5.28)$$

In order to estimate $\tilde{\xi}(q_0, 1)$ and $\tilde{\xi}_q(q_0, 1)$, we write the system (5.4), (5.5) in the following form:

$$\begin{aligned} \left[\frac{w_p}{H_p^3} \right]_p + \left[\frac{w_q}{H_p} \right]_q &= [\mathcal{N}_1^*(w, f)]_p + [\mathcal{N}_2^*(f)]_q, \quad (q, p) \in S, \\ \frac{w_p}{H_p^3} - w &= \mathcal{N}_1^*(w, f) \quad \text{when } p = 1, \quad w = 0 \quad \text{when } p = 0. \end{aligned}$$

Here

$$\mathcal{N}_1^*(w, f) = \frac{w_p^3}{H_p^3 h_p^2} + \frac{3}{2} \frac{w_p^2}{H_p^2 h_p^2} + \frac{f^2}{2} \quad \text{and} \quad \mathcal{N}_2^*(f) = \frac{f w_p}{H_p}.$$

Comparing this and the system (5.12), (5.13), we obtain that $\tilde{w} = (I - \mathcal{P}_1)(w)$ must satisfy the problem

$$\begin{aligned} \left[\frac{\tilde{w}_p}{H_p^3} \right]_p + \left[\frac{\tilde{w}_q}{H_p} \right]_q &= [\mathcal{N}_1^*]_p + [\mathcal{N}_2^*]_q + \frac{\phi_0}{H_p} \int_0^1 (\mathcal{N}_1^* \phi'_0 - [\mathcal{N}_2^* \phi_0]_q) \, dp, \\ \frac{\tilde{w}_p}{H_p^3} - \tilde{w} &= \mathcal{N}_1^* \quad \text{when } p = 1, \quad \tilde{w} = 0 \quad \text{when } p = 0. \end{aligned}$$

Let $\tilde{w}^{(1)}$ and $\tilde{w}^{(2)}$ correspond to $\eta^{(1)}$ and $\eta^{(2)}$, respectively. Since both $\tilde{w}^{(1)}$ and $\tilde{w}^{(2)}$ solve the last problem, but with different right-hand side terms, their difference, which we also denote by \tilde{w} , must satisfy the following problem:

$$\begin{aligned} \left[\frac{\tilde{\xi}_p}{H_p^3} \right]_p + \left[\frac{\tilde{\xi}_q}{H_p} \right]_q &= [\mathcal{J}_1^*]_p + [\mathcal{J}_2^*]_q + \frac{\phi_0}{H_p} \int_0^1 (\mathcal{J}_1^* \phi'_0 - [\mathcal{J}_2^* \phi_0]_q) \, dp \\ \frac{\tilde{\xi}_p}{H_p^3} - \tilde{\xi} &= \mathcal{J}_1^* \quad \text{when } p = 1, \quad \tilde{\xi} = 0 \quad \text{when } p = 0. \end{aligned} \tag{5.29}$$

Here

$$\mathcal{J}_1^* = \mathcal{N}_1^* \left(w^{(1)}, f^{(1)} \right) - \mathcal{N}_1^* \left(w^{(2)}, f^{(2)} \right), \quad \mathcal{J}_2^* = \mathcal{N}_2^* \left(w^{(1)}, f^{(1)} \right) - \mathcal{N}_2^* \left(w^{(2)}, f^{(2)} \right).$$

Similarly, the function ξ must satisfy

$$\begin{aligned} \left[\frac{\xi_p}{H_p^3} \right]_p + \left[\frac{\xi_q}{H_p} \right]_q &= [\mathcal{J}_1^*]_p + [\mathcal{J}_2^*]_q, \\ \frac{\xi_p}{H_p^3} - \xi &= \mathcal{J}_1^* \quad \text{when } p = 1, \quad \xi = 0 \quad \text{when } p = 0. \end{aligned} \tag{5.30}$$

Let $\theta = \theta'_0 = \theta''_0$ and $\epsilon_1 = \min\{\epsilon'_0, \epsilon''_0\}$, where θ'_0 (θ''_0) and ϵ'_0 (ϵ''_0) are θ_0 and ϵ_0 , respectively, that exist according to Lemma 5.5 (5.6, respectively). All of them depend only on ω_0 . Without loss of generality, we assume that

$$\epsilon = \max_{i=1,2} \left\{ \|w^{(i)}\|_{C^2(\overline{S})} + \|f^{(i)}\|_{C^1(\overline{S})} \right\} \leq \min\{1, \epsilon_1\}.$$

First, let us show that

$$\int_{\mathbb{R}} \|\xi\|_{C^{1,\alpha}(D_t)}^2 e^{-\theta|t-q_0|} \, dt \leq C(\omega_0) \int_{\mathbb{R}} \|\xi\|_{L^2(2D_t)}^2 e^{-\theta|t-q_0|} \, dt, \tag{5.31}$$

where $D_t = [t-1, t+1] \times [0, 1]$, $2D_t = [t-2, t+2] \times [0, 1]$ and $\alpha = 1/2$ (the latter can be any number between 0 and 1). Let us apply Theorem 9.3, [1], to the systems (5.29) and (5.30). In the case of (5.30), this theorem gives

$$\|\xi\|_{C^{1,\alpha}(D_t)} \leq C(\omega_0) \left[\|\mathcal{J}_1\|_{C^\alpha(2D_t)} + \|\mathcal{J}_2\|_{C^\alpha(2D_t)} + \|\xi\|_{L^2(2D_t)} \right].$$

It follows from the definition of \mathcal{J}_i^* , $i = 1, 2$, that

$$\|\xi\|_{C^{1,\alpha}(D_t)} \leq C(\omega_0) [\epsilon \|\xi\|_{C^{1,\alpha}(2D_t)} + \|\xi\|_{L^2(2D_t)}]. \quad (5.32)$$

After squaring (5.32) and multiplying the result by $e^{-\theta|t-q_0|}$, we integrate over \mathbb{R} , thus obtaining

$$\begin{aligned} \int_{\mathbb{R}} \|\xi\|_{C^{1,\alpha}(D_t)}^2 e^{-\theta|t-q_0|} dt &\leq \epsilon C(\omega_0) \int_{\mathbb{R}} \|\xi\|_{C^{1,\alpha}(2D_t)}^2 e^{-\theta|t-q_0|} dt \\ &\quad + C(\omega_0) \int_{\mathbb{R}} \|\xi\|_{L^2(2D_t)}^2 e^{-\theta|t-q_0|} dt = I_1 + I_2. \end{aligned} \quad (5.33)$$

Furthermore, we have that

$$I_1 \leq C'(\omega_0) \epsilon \int_{\mathbb{R}} \|\xi\|_{C^{1,\alpha}(D_t)}^2 e^{-\theta|t-q_0|} dt, \quad (5.34)$$

which is a consequence of the inequality

$$\|\xi\|_{C^{1,\alpha}(2D_t)}^2 \leq 2 \left[\|\xi\|_{C^{1,\alpha}(D_{t-1})}^2 + \|\xi\|_{C^{1,\alpha}(D_t)}^2 + \|\xi\|_{C^{1,\alpha}(D_{t+1})}^2 \right].$$

Let $\epsilon < \epsilon_2 = [C']^{-1}/2$, where C' is the constant in (5.34). Then (5.33) and (5.34) imply (5.31).

On the other hand, applying Theorem 9.3, [1], to (5.29) and using the definition of \mathcal{J}_i^* , $i = 1, 2$, we obtain

$$\|\tilde{\xi}\|_{C^{1,\alpha}(D_t)} \leq C(\omega_0) [\epsilon \|\xi\|_{C^{1,\alpha}(2D_t)} + \|\tilde{\xi}\|_{L^2(2D_t)}]. \quad (5.35)$$

Again we square this, multiply by $e^{-\theta|t-q_0|}$ and integrate the result over \mathbb{R} , thus obtaining

$$\begin{aligned} \int_{\mathbb{R}} \|\tilde{\xi}\|_{C^{1,\alpha}(D_t)}^2 e^{-\theta|t-q_0|} dt &\leq \epsilon C(\omega_0) \int_{\mathbb{R}} \|\xi\|_{C^{1,\alpha}(2D_t)}^2 e^{-\theta|t-q_0|} dt \\ &\quad + C(\omega_0) \int_{\mathbb{R}} \|\tilde{\xi}\|_{L^2(2D_t)}^2 e^{-\theta|t-q_0|} dt. \end{aligned}$$

It follows from (5.34) and (5.31) that the right-hand side is less than or equal to

$$C(\omega_0) \epsilon \int_{\mathbb{R}} \|\xi\|_{L^2(2D_t)}^2 e^{-\theta|t-q_0|} dt + C(\omega_0) \int_{\mathbb{R}} \|\tilde{\xi}\|_{L^2(2D_t)}^2 e^{-\theta|t-q_0|} dt.$$

Changing the order of integration, we apply Lemmas 5.5 and 5.6, thus estimating these integrals as follows:

$$\int_{\mathbb{R}} \|\tilde{\xi}\|_{C^{1,\alpha}(D_t)}^2 e^{-\theta|t-q_0|} dt \leq \epsilon C''(\omega_0) [\xi_0^2(q_0) + \xi_0'^2(q_0)].$$

Combining this inequality and (5.28), we obtain

$$\xi_0^2(q_0) + \xi_0'^2(q_0) \leq C(\omega_0) [\xi(q_0, 1)^2 + \xi'(q_0, 1)^2] + \epsilon C'''(\omega_0) [\xi_0^2(q_0) + \xi_0'^2(q_0)].$$

If $\epsilon < \epsilon_0 = \min\{\epsilon_1, \epsilon_2, [2C'''(\omega_1)]^{-1/\gamma}\}$, then we get the required inequality. \square

5.2.2 Proof of Theorem 5.1

Now we are in a position to complete the proof of Theorem 5.1. Let us denote by ϵ'_0 , ϵ''_0 and ϵ'''_0 the constant ϵ_0 existing according to Lemma 5.5, 5.6 and 5.7, respectively. Let $\epsilon_0 = \min\{\epsilon'_0, \epsilon''_0, \epsilon'''_0\}$, whereas θ_0 is the constant existing by Lemmas 5.5 and 5.6; both ϵ_0 and θ_0 depend only on ω_0 . Then assertion (a) of Theorem 4.1 and Lemma 5.1 allow us to find r'' depending on M and ω_1 so that the inequality

$$\max_{i=1,2} \left\{ \|w^{(i)}\|_{C^2(\overline{S})} + \|f^{(i)}\|_{C^1(\overline{S})} \right\} \leq \epsilon_0$$

holds for every solution of problem P_r^M provided $r \in (r_c, r'']$, where r'' exists by Theorem 5.1. Then we apply Lemmas 5.5, 5.6 and 5.7, thus obtaining the inequality

$$\begin{aligned} & \|w_p^{(1)} - w_p^{(2)}\|_{\theta, q_0} + \|f^{(1)} - f^{(2)}\|_{\theta, q_0} \\ & \leq C(M, \omega_1) \left[|w^{(1)}(q_0, 1) - w^{(2)}(q_0, 1)|^2 + |w_q^{(1)}(q_0, 1) - w_q^{(2)}(q_0, 1)|^2 \right], \end{aligned}$$

from which (5.1) follows. Thus, the proof of Theorem 5.1 is complete.

6 Proof of Theorem 2.1 and Verification of the Benjamin–Lighthill Conjecture

Theorem 2.1 is a consequence of the following three facts. First, in view of Lemma 5.1 all solutions of the problem P_r^M are of small amplitude when r belongs to $(r_c, r_*]$, where r_* depends only on M and ω_1 . Second, according to results obtained in [9] all small amplitude waves are exhausted by a continuous branch of Stokes waves bifurcating from a horizontal shear flow and terminating by the solitary wave of elevation. Third, Theorem 2.2 implies that these solutions are uniquely parametrized by their height at the crest provided the latter lies on the y -axis.

Now we turn to verification of the Benjamin–Lighthill conjecture for near-critical values of Bernoulli's constant. Namely, we suppose that $r \in (r_c, r_*]$, where $r_* < \min\{r', r''\}$ and $r', r'' \in (r_c, r_0)$ are the values that exist according to Theorems 2.1 and 2.2, respectively.

Theorem 2.1 says that Stokes-wave solutions of problem P_r^M are parameterized by their heights at the crest provided the latter is located on the y -axis. Let $(\psi^{(t)}, \eta^{(t)})$ be such a solution for some $t \in (d_+(r), \eta^{(s)}(0))$ (we recall that $\eta^{(s)}(0)$ is the height of the corresponding solitary wave at its crest). Since the flow force does not depend on x , its value for $(\psi^{(t)}, \eta^{(t)})$ is as follows:

$$s(t) = \left[r + \frac{2}{3}\Omega(1) \right] t - \frac{1}{3} \left\{ t^2 + \int_0^t \left[[\psi_x^{(t)}]_{x=0}^2 - [\psi_y^{(t)}]_{x=0}^2 + 2\Omega(\psi^{(t)}(0, y)) \right] dy \right\}.$$

Let us show that *this function strictly decreases on $(d_+(r), \eta^{(s)}(0))$.*

For this purpose we write s in terms of the function $h(q, p; t)$ that corresponds to $(\psi^{(t)}, \eta^{(t)})$ through the partial hodograph transform, thus obtaining

$$3s(t) = [3r + 2\Omega(1)]t - t^2 + \int_0^1 \left[\frac{1}{h_p^2(0, p; t)} - 2\Omega(p) \right] h_p(0, p; t) dp.$$

Denoting differentiation with respect to t by the top dot, we get from the previous equality that

$$3\dot{s}(t) = 3r + 2\Omega(1) - 2t - \int_0^1 \left[\frac{\dot{h}_p}{h_p^2} + 2\Omega(p)\dot{h}_p \right]_{q=0} dp.$$

Integrating by parts, we obtain

$$3\dot{s}(t) = 3r + 2\Omega(1) - 2t - \dot{h}(0, 1; t) \left[\frac{1}{h_p^2(0, 1; t)} + 2\Omega(p) \right] + 2 \int_0^1 \left[\frac{h_{qq}}{h_p} \dot{h} \right]_{q=0} dp.$$

Here the equation (3.1) is taken into account to simplify the integrand. Since $h(0, 1; t) = t$, the Bernoulli equation (3.3) gives that

$$\dot{s}(t) = \frac{3}{2} \int_0^1 \left[\frac{h_{qq}}{h_p} \dot{h} \right]_{q=0} dp.$$

Indeed, the out of integral terms cancel because the point $(q, p) = (0, 1)$ corresponds to a wave crest.

To complete the proof of our assertion, let us show that $h_{qq}(0, p; t) < 0$ and $\dot{h}(0, p; t) > 0$ for all $p \in (0, 1)$ and $t \in (d_+(r), \eta^{(s)}(0))$.

In order to prove the first of these inequalities, we denote by 2Λ the wavelength of the Stokes wave described by $h(q, p; t)$ with some $t \in (d_+(r), \eta^{(s)}(0))$. Then we have that

$$h_q(0, p) = h_q(\Lambda, p) = 0 \quad \text{for all } p \in [0, 1] \quad \text{and} \quad h_q(q, 0) = 0 \quad \text{for all } q \in [0, \Lambda].$$

On the other hand, the inequality $h_q(q, 1) < 0$ holds for all $q \in (0, \Lambda)$. These properties of h_q allow us to apply the maximum principle to this function in the rectangle $(0, \Lambda) \times (0, 1)$, which yields the required inequality for $h_{qq}(0, p; t)$.

Instead of proving the inequality $\dot{h}(0, p; t) > 0$, let us show that $h(0, p; t)$ is an increasing function of t on $(d_+(r), \eta^{(s)}(0))$. Putting

$$\xi(q, p) = h(q, p; t_1) - h(q, p; t_2), \quad t_1, t_2 \in (d_+(r), \eta^{(s)}(0)),$$

we combine the inequality (5.31) and Lemmas 5.5, 5.6, 5.7, where $q_0 = 0$, which gives

$$\|\xi\|_{C^{1,\alpha}([-1,1] \times [0,1])} \leq C(\omega_0)|t_1 - t_2|.$$

This and (5.35) imply that

$$\|\tilde{\xi}\|_{C^{1,\alpha}([-1,1] \times [0,1])} \leq \epsilon C(\omega_0)|t_1 - t_2|,$$

where

$$\epsilon = \max_{i=1,2} \left\{ \|w^{(i)}\|_{C^2(\overline{\mathcal{S}})} + \|f^{(i)}\|_{C^1(\overline{\mathcal{S}})} \right\}.$$

The last inequality yields that

$$\left| \tilde{\xi}(0, p) - \frac{\tilde{\xi}(0, 1)}{\phi_0(1)} \phi_0(p) \right| / |t_2 - t_1| < \epsilon C(\omega_0).$$

Finally, we can write

$$\xi(0, p) = \xi_0(0) \phi_0(p) + \tilde{\xi}(0, p) = \xi(0, 1) \frac{\phi_0(p)}{\phi_0(1)} + \left[\tilde{\xi}(0, p) - \frac{\tilde{\xi}(0, 1)}{\phi_0(1)} \phi_0(p) \right].$$

Thus, if ϵ is small enough, then the obtained inequalities give that

$$[h(0, p; t_1) - h(0, p; t_2)] / (t_1 - t_2) > \frac{\phi_0(p)}{2\phi_0(1)} \quad \text{for all } p \in (0, 1),$$

which completes the proof of our assertion and, consequently, verification of the Benjamin–Lighthill conjecture.

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